THE PROGRESSIVE EUCLID

BOOKS I-II

A.T. RICHARDSON

THE "PROGRESSIVE" EUCLID

BOOKS I. AND II.

WITH NOTES, EXERCISES, AND DEDUCTIONS

EDITED BY

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PREFACE.

The present edition of Euclid is written in the interests of beginners, and especially of the dull boys to whom Euclid is often one of the most distasteful subjects of the School Curriculum.

As many difficulties arise from boys having to reason about things of which they have no very clear conception, a series of questions has been given, some 230 in number, before the Propositions begin, arranged in groups to correspond with the pages containing the Definitions, etc. They will be found very simple, for their aim is to encourage boys to think, and not to puzzle them. Similar questions, to serve as tests to see whether the reasoning is understood, are given after each Proposition; these, and the notes, being intended to supplement and test oral teaching.

An attempt has been made to simplify the language as far as possible, while keeping to the original order and arrangement; and by using simpler words in the earlier propositions, to familiarize gradually with Euclidean language. In particular, the word "Hypothesis" has been restricted to mean an *incorrect supposition.* As a supposition is generally something distinct from a fact, it gives an air of unreality to a Proposition, to call the part "given" a Hypothesis, and especially when the same word is used, later on, to denote an impossible supposition. It is easier for boys afterwards to class the two ideas together in one word, than, as beginners, to differentiate

between two very diverse meanings of the same word. Symbols are introduced gradually, that learners may devote all their thought to the reasoning in the early Propositions, without the distraction of having to decipher hieroglyphics. In the later Propositions they are used freely; for, when a boy has become accustomed to them, the eye takes in the argument more rapidly, and more completely. And, since the written word is itself only a *symbol* for the spoken word, it is difficult to see what is the objection to making use of a clearer and shorter symbol. No purely Algebraic symbols are used.

The proof is arranged so as to teach boys to write out neatly and quickly, and in a good form for an Examiner; and it is broken up into sections, that a boy may take in one piece at a time. The figures are placed between the two parts of the Particular Enunciation, so that a clear break may be made between what is given, and what is to be proved. The whole of each Proposition, except in two cases, is in view at the same time.

Constructions are shown by dotted lines, and the more important lines of the figure are thickened. In some cases, letters other than the usual ones are used, to accustom the beginners to the use of them; and use has been made of this to connect more closely Prop. IV. with its first application in Prop. V. The applications of Props. IV. and VIII. are so arranged, that boys may learn to write down first the 3 elements of one triangle, and then the 3 elements of the other; but at the same time make the two sides of the equality correspond.

The Corollary to Prop. XI. is omitted. The Theorem which it seeks to prove is, at least, as axiomatic as Ax. 10; but, more than that, it has been tacitly assumed previously, in Def. 10, Ax. 11, Post. 2, Props. V. and VII., and in the Construction of the Corollary itself!

In Book II. beginners are often bewildered with the "backwards and forwards" arrangement of the proof, by the difficulty of remembering how it begins, and by the fact that the chain of reasoning is broken at every step. To avoid these, the proof always begins with one side of the equality to be proved, and Axiom 1 is assumed at every line, without so much of the unnecessary repetition. In this Book, the connexion of the Propositions with the well-known Algebraic formulæ of two dimensions is pointed out; for Algebra, with its concise forms, gives at a glance a far clearer idea of the meaning of the Proposition as a whole, than the necessarily verbose Geometrical method.

The edition followed has been Todhunter's, and to his notes I owe much, including the alternative proofs of Props. 5, 8, 29 of Book I. Alternative proofs are also given for Props. 13, 24, 26 and 32, Cor. I. of Book I., and Props. 4—10 of Book II. All these are given *in full*.

A collection of "Riders" collected from various sources is given at the end of the volume, divided into 3 parts, according to their difficulty.

Some well-known Problems are also worked out, for the benefit of the higher classes.

Any suggestions or corrections will be very gladly received.

A. T. RICHARDSON.

I. W. COLL., RYDE,
Jany. 1891.

EUCLID'S ELEMENTS.

BOOK I.

DEFINITIONS.

1. **A Point** is that which has position, but has no size.

2. **A Line** is length without breadth.

3. The extremities of a line are points.

4. **A Straight Line** is that which lies evenly between its extreme points.

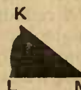
5. **A Superficies** (or Surface) is that which has only length and breadth.

6. The extremities of a superficies are lines.

NOTES.

1. The conceptions of a point without size, or a line without breadth, may be best understood by looking at the corners and edges of a well-finished cubical block. It will then be easily seen that no definite *size* can be ascribed to the corners, or *breadth* to the edges.

A point is generally denoted by a letter, as "the point M."



2. A line is denoted by the two letters which are at its ends; as "the line KL." (In this figure the 'lines' or *edges* have no breadth.)

3. Notice that the 'end' of a line cannot have size. If we cut off ever so small a part of the line, we cannot be yet at the end.

4. A straight line is sometimes called a **right line.**

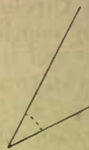
5. A surface necessarily presupposes the existence of a solid (having *thickness*), to which it belongs. But the *surface* has no thickness, since however thin a slice we take off, it will have *two* surfaces, one on each side; and we should be *below* the surface.

The best example of a superficies is a shadow.

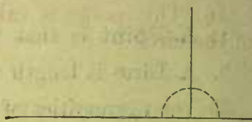
7. **A Plane Superficies** (or Flat Surface) is that in which any two points being taken, the straight line between them lies wholly in that superficies.

8. **A Plane Angle** is the inclination of two lines to each other in a plane, which meet together, but are not in the same direction.

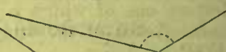
9. **A Plane Rectilineal Angle** is the inclination of two *straight lines* to each other, which meet together, but are not in the same *straight line.*



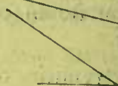
10. When one straight line standing on another straight line makes the adjacent angles equal to one another, each of these angles is called a **Right Angle**, and each straight line is said to be **Perpendicular** to the other.



11. An **Obtuse Angle** is greater than a right angle.



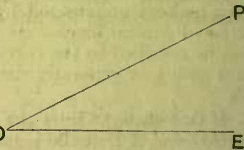
12. An **Acute Angle** is less than a right angle.



NOTES.

7. This definition is practically used by a carpenter, when he uses a "straight edge" to try if a piece of wood is quite flat.

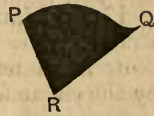
9. The idea of an angle is best obtained by imagining two lines *OE*, *OP*, hinged at *O*, so that *OP* can revolve round *O*, whilst *OE* remains fixed. As *OP* revolves, the angle becomes larger or smaller, until *OP* comes to be in the same straight line with *OE*, when the "angle" (in Euclid's sense) ceases to exist.



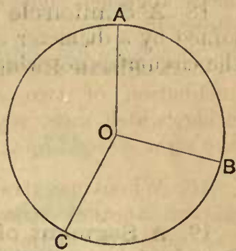
Or, suppose *EO* and *OP* to represent two *roads;* then the angle is the "turn" which a man has to make when he goes from one along the other. In the figure above he will evidently have to make a very "sharp" turn (or "acute" angle) to the right or left hand.

It is easy to see that the *distance* he has to go along the roads makes no difference to the sharpness, or obtuseness, of the turn; and so the *length* of the straight lines which form the angle makes no difference to the size of the angle.

13. A term or boundary is the extremity of anything.



14. A **Figure** is that which is enclosed by one or more boundaries.

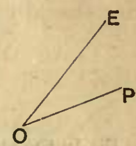


15. A **Circle** is a plane figure contained by one line called the **Circumference**, and is such that all straight lines drawn *from* a certain point within the figure, *to* the circumference, are equal to each other.

16. This point is called the **Centre** of the circle.

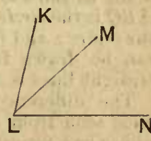
NOTES.

The point, where the two lines which form the angle meet, is called the *vertex*.

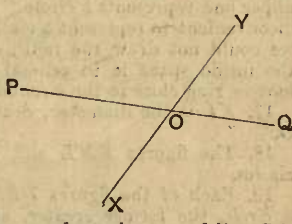


An angle is sometimes denoted by the letter at its vertex, as "the angle O"; or, more frequently, by *three* letters, one of which is at the vertex, and the others, one on each of the arms of the angle, as "the angle EOP."

The letter at the vertex must be the middle one of the three.



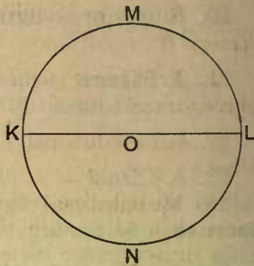
When one straight line forms an arm of each of two angles, these angles are said to be *adjacent* to each other. Thus the angles KLM and MLN are adjacent, because the line LM belongs to both.



If two straight lines cut one another, as PQ and XY at the point O, then the angles XOP, YOQ, are said to be *vertically opposite* to each other, and so are POY and XOQ.

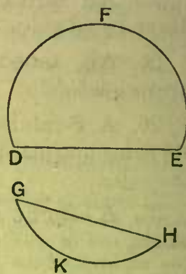
14. The figure PQR has for boundaries, or edges, the curved line PQ, and the straight lines PR, and RQ. It should be noticed that no *breadth* can be ascribed to these edges.

17. **A Diameter** of a circle is a straight line drawn through the centre, and terminated at both ends by the circumference.



18. **A Semicircle** is a figure contained by a diameter and the part of the circumference which it cuts off.

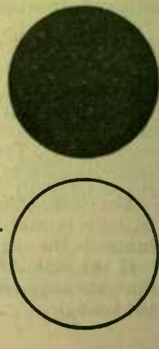
19. **A Segment of a Circle** is a figure contained by a straight line, and the part of the circumference which it cuts off.



NOTES.

15. In the above figure the *space within* the outside edge of the ring *ABC* is the *circle;* the *outside edge* of the ring is the *circumference;* *O* is the *centre;* and *OA*, *OB*, *OC*, are some of the equal straight lines which can be drawn from the centre to the circumference. Each of these straight lines is called a *radius*.

The difference between a circle and a ring is at once seen from the accompanying figures, where the upper one represents a circle. But it would be very inconvenient to represent a circle in this way, because we could not draw the radii, etc. Consequently all the inside space is in general left clear, except the narrow ring close to the circumference.



17. *KL* is the diameter, drawn through the centre *O*.

18. The figures *KML* and *KNL* are each semicircles.

19. Each of the figures *DEF*, *GHK* is a *segment of a circle*, the former greater, and the latter smaller, than a semicircle. The straight lines, *DE*, *GH*, are called *chords* of the circle. The "parts of the circumferences," *DFE*, *GHK*, are called *arcs*.

20. **Rectilineal figures** are those which are contained by *straight* (or *right*) lines.

21. **Trilateral** figures, or **Triangles,** are contained by three straight lines.

22. **Quadrilateral** figures are contained by four straight lines.

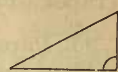
23. **Multilateral** figures, or **Polygons,** are contained by more than four straight lines.

24. **An Equilateral Triangle** has three equal sides.

25. **An Isosceles Triangle** has two equal sides.

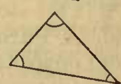
26. **A Scalene Triangle** has all its sides unequal to each other.

27. **A Right-Angled Triangle** has a right angle.



28. **An Obtuse-Angled Triangle** has an obtuse angle.

29. **An Acute-Angled Triangle** has *three* acute angles.

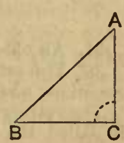


NOTES.

20. Figures contained by curved lines are called *curvilinear.*

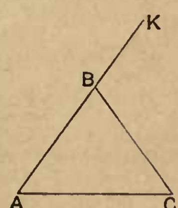
27. In a right-angled triangle the side which is opposite to the right triangle is called the *hypoteneuse.*

(In the figure AB is the hypoteneuse.)

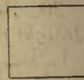


The side which is opposite to an angle of a triangle is said to **subtend** that angle; thus, BC subtends the angle A.

If the side AB of a triangle be produced to K, then the angle KBC is called an **exterior angle** of the triangle.



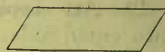
30. A **Square** is a four-sided figure which has all its sides equal, and all its angles right angles.



31. An **Oblong** is a four-sided figure which has not all its sides equal, but all its angles are right angles.

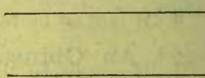
32. A **Rhombus** is a four-sided figure which has all its sides equal, but its angles are not right angles.

33. A **Rhomboid** is a four-sided figure which has its opposite sides equal, but all its sides are not equal, nor its angles right angles.

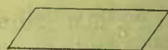


34. All other four-sided figures are called **Trapeziums.**

35. **Parallel straight lines** are such as lie in the same plane, and which being produced ever so far both ways, do not meet.



36. A **Parallelogram** is a four-sided figure whose opposite sides are parallel.



Notes.

31. An oblong is more frequently called a **rectangle.**

36. The term parallelogram will be seen to include the square, oblong, rhombus, and rhomboid.

The straight line joining two opposite angles of a parallelogram is called the **diameter** or **diagonal.**

POSTULATES.

1. Let it be granted that a straight line may be drawn from any one point to any other point.

2. Let it be granted that a terminated straight line may be produced to any length in that straight line.

3. Let it be granted that a circle may be described with any centre at any distance from that centre.

Note.

These postulates are "Requests" that these three simple problems may be assumed as possible. In other words, they amount to a supposition that the student has a *ruler* and *a pair of compasses*. These, however, are not supposed to be used for measuring.

AXIOMS.

1. Things which are equal to the same thing are equal to one another.

2. If equals be added to equals the wholes are equal. (Addition.)

3. If equals be taken from equals the remainders are equal. (Subtraction.)

4. If equals be added to unequals the wholes are unequal. (Addition.)

5. If equals be taken from unequals the remainders are unequal. (Subtraction.)

6. Things which are double of the same thing are equal. (Multiplication.)

7. Things which are halves of the same thing are equal. (Division.)

8. Magnitudes which coincide with one another—(that is, which can fit exactly on one another)—are equal.

9. The whole is greater than its part.

10. Two straight lines cannot enclose space.

11. All right angles are equal to one another.

12. See Prop. 17.

Note.

Axioms are "*self-evident truths*," which are at once seen to require no proof. Euclid calls them "Common Notions."

EXERCISES.

I.

1. Draw a figure which will have five points.
2. How many lines are there to the figure you have drawn?
3. Mention six examples of a "Plane Superficies," and six examples of a "superficies" which is not "plane."
4. Explain what is the meaning of the word "plane."
5. Draw a figure which will show a point without size and a line without breadth.

6. Is this a line?  Give a reason for your answer.

FIG. 1.

7. Are the edges of Fig. 2 lines? Are they straight lines?

FIG. 2.

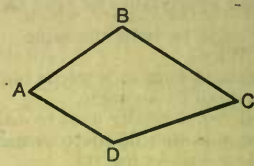


FIG. 3.

8. Mention *exactly* which are the lines of Fig. 3.
9. Is the mark on the paper between A and B a line? Give your reason.
10. Is the figure $ABCD$ a Plane Superficies? Why?

II.

1. Explain what is meant in Def. 7 by "the straight line lies wholly in the superficies."
2. If the straight line between two points does *not* lie wholly in the superficies, what will it look like? What would you then say about the superficies?
3. Could you take two points on the surface of a polished round ruler, so that the straight line between them lies wholly in the surface? Is the surface a "plane superficies"?
4. How many surfaces has a round ruler? a brick? an india-rubber ball? Are any of these "plane superficies"? Give reasons.
5. How many edges are there in each of the above objects? Are any of the edges "straight lines"? Why?
6. Explain why a superficies has no thickness?
7. Has a line thickness? What has it?
8. Draw a figure to represent a straight line. Is what you have drawn a straight line? Where is the straight line really? How many straight lines are there in what you have drawn?

9. Make two other figures to represent
   (i.) A straight line about twice as long as the first.
   (ii.) A straight line about four times as long as the first.
10. Is a piece of paper a "Plane Superficies"?

III.

1. Draw an "obtuse plane rectilineal angle."
2. Draw three plane angles, which will not be rectilineal.
3. Draw an acute angle.
4. Make one of the lines of the angle you have just drawn longer, and the other one shorter. Is the angle now larger or smaller than it was?
5. Which is the larger of these two angles? *A* or *B*? Why?
6. Must the two lines which make an angle be both the same length? Why?

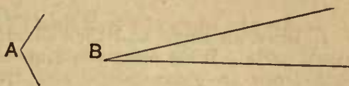


Fig. 4.

7. Explain in your own words (not the words of the definition) what you understand by an angle.
8. What is practically used to make a right angle?
9. Draw a sketch of the tool used by a carpenter to make a right angle. How does he use it?
10. Name the angles in these figures below by the letters :—

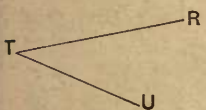


Fig. 5.

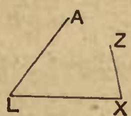


Fig. 6.

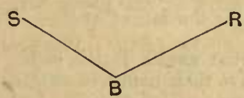


Fig. 7.

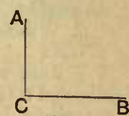


Fig. 8.

IV.

1. Make a copy of the accompanying figure, and on it mark the following angles by putting a dotted line round the vertex of the angle, similar to that in the figures of Definitions 11 and 12. Number your angles.
   1. *AEB*. 2. *BAE*. 3. *ABE*. 4. *ABC*.
   5. *ECB*. 6. *ACB*. 7. *ADB*. 8. *ADE*.

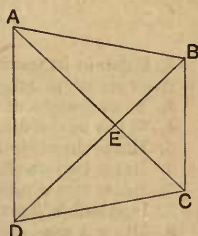


Fig. 9.

2. How is an angle made larger? How could you show this practically with a pocket knife? What "Mathematical Instrument" would also show it?
3. Which is the larger angle :—(i.) *BAC* or *BAD*? (ii.) *DBA* or *CBA*? Give your reason.

4. Draw an acute angle, and then make one twice as large, as nearly as you can.
5. Draw an obtuse angle, and then draw a line to divide it into two angles.
6. Draw a right angle, either with a "set square," or as nearly as you can make it, and divide it into three angles.
7. What is the vertex of (i.) the angle AED; (ii.) the angle BCA; (iii.) the angle DBA (in the figure above); (iv.) the angle XYZ?
8. Draw a figure to represent the last angle in question 7.
9. Of what kind is the angle you have drawn?
10. Draw two straight lines as nearly perpendicular to one another as you can.

V.

1. Draw an angle, and then divide it as nearly as you can into two equal parts. What must you draw in order to do this?
2. Draw an obtuse angle and divide it into two parts which are not equal. Say which is the larger of them, and why you think so.
3. In how many ways can an angle be named? Illustrate by giving all the different ways of naming the angle with the dotted line round it in this figure.
4. Of what kind are the angles WXZ, XVW, YZX?
5. What is the symbol used to denote an angle?
6. Which is the larger angle—VWX or XWZ?
7. Into what angles is the angle YZW divided? Give their names in as many ways as can be done.
8. Name all the angles which have the vertex W.
9. What angle do we get by adding the angles VZY and WZX?
10. If we take the angle XWV from the angle XWZ what is left?

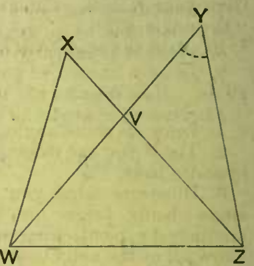


Fig. 10.

VI.

1. Explain in your own words what is a figure?
2. Is an angle a figure? Give your reason.
3. Is a segment of a circle a figure? Why?
4. What is the difference between a circle, a ring, a circumference?
5. Draw three circles which all have the same centre.
6. Draw two circles with different centres but *the same* radius.
7. Draw two circles with different centres, but with *equal* radii.
8. Is an arc a figure? Give reasons.
9. Draw a semi-circle.
10. Is a semi-circle a segment of a circle? Is a segment of a circle a semi-circle?

VII.

1. Explain why the length of the arms of an angle makes no difference to the size of the angle.
2. Name the angle which is adjacent to the angle RVM in this figure.
3. There are eight ways of designating the angle RVO. Find them.
4. Which is the larger angle—MQO or KQR? Why?
5. What is the angle vertically opposite to ROV?
6. What other angle in the figure has an angle vertically opposite to it?
7. What is the sum of the angles RMO and VMQ?
8. If we take the angle RMV from the angle KMV, what is the remainder?
9. Can the angle VOQ be called "the angle O"? Give your reason. Can the angle RKM be called "the angle K"?
10. Add the angle KMR to the angle RMO.

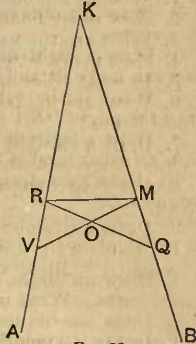


FIG. 11.

VIII.

1. Draw an acute angle, with an obtuse angle adjacent to it.
2. Try if you can draw an acute angle with an obtuse angle vertically opposite to it.
3. Illustrate what you mean by one angle being larger or smaller than another, by mentioning any roads you know of, in the neighbourhood.
4. Is the angle LMQ larger or smaller than the angle LMZ?
5. Take away the angle YLQ from the angle MLY. What is left?
6. What do we get by adding the angle LQM to the angle MQK?
7. Which of the following pairs of angles can be added together?
   (i.) KLM and QLM.
   (ii.) LMZ and LKM.
   (iii.) YLQ and MLQ.
   (iv.) LQM and QMK.

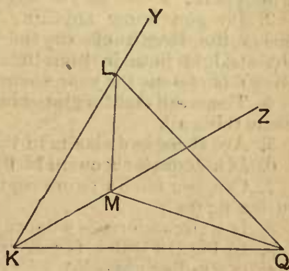


FIG. 12.

Give the resulting angles, where they can be added together.

8. Of what kind are the angles LKM, KMQ, MQL?
9. Is the angle YKZ larger or smaller than the angle LKM? Give a reason for your answer.
10. What angles are adjacent to the angles KLM and LMZ respectively?

IX.

1. What is the boundary of a circle called?
2. How many boundaries has a segment of a circle?
3. What are the names given to them?
4. How many letters are needed to name an angle?
5. In what order must they be put?
6. What are the boundaries of (i.) a line, (ii.) a point, (iii.) a rectilineal figure?
7. Draw a straight line, and with one end as centre, and the line as radius, describe a circle.
8. Draw two angles vertically opposite to one another, and name them.
9. Draw a straight line standing on another straight line.
10. Explain the difference between "a figure" and "the boundaries of a figure." Illustrate your answer by the case of a field.

X.

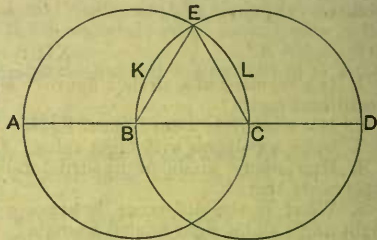


FIG. 13.

1. In Fig. 13 mention :—
   (i.) The names of the two circles.
   (ii.) The centres of both circles.
   (iii.) All the radii of the left-hand circle.
2. Is there a diameter to either of these circles? If so, which is it?
3. Do you know anything about the length of any of the straight lines in Fig. 13? Give *full* reasons for your answer.
4. There are eight segments of circles in the above figure. Try and name them all.
5. Are there any chords in the figure?
6. Is a diameter a chord? Is a chord a diameter?
7. Can you have a figure contained by one straight line? Is a circle such a figure?
8. Is a circumference a figure? Give your reason. If not, what is it?
9. In Fig. 13 which is the radius which is *common* to both circles? (*i.e.*, is a radius of both.)
10. How do you know the radius of a circle when you see it?

XI.

1. Draw two circles which cut one another.
2. Which two points are on the circumferences of both these circles? Are there any more such points?
3. Try if you can draw two circles which will cut one another at more than two points.

4. Draw two circles which cut one another, and then another circle through the points where the first two circles cut?

5. We call these points "(——) to the three circles." Supply the word which is wanting.

6. Draw two segments of a circle, one greater and one less than a semi-circle.

7. Is Fig. 14, strictly speaking, an arc? Give your reason.

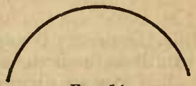


Fig. 14.

8. Draw a circle, with any radius. With centre on the circumference of this circle, and *the same radius*, describe another circle. With centres at the points where this circle cuts the first circle, and *the same radius*, describe other circles. Continue this all round the first circle.

9. Draw straight lines all round the inner circle from point to point, where the outer circles cut it. Do you know anything about the length of the sides of the rectilineal figure you have thus made?

10. Draw four circles which shall all pass through the same two points.

XII.

1. Is a segment of a circle a figure? Is it a plane figure? Is it a plane rectilineal figure?

2. Is a semicircle a curvilinear figure?

3. Draw a polygon with eight sides. How many angles has it?

4. How many kinds of quadrilateral figures are there?

5. What is the difference between a right angle and a right-angled triangle?

6. What are the different sorts of triangles, classed according to their sides?

7. Give the names of all the triangles in Fig. 15.

8. What is the difference between "the angle *EHF*" and "the triangle *EHF*"?

9. What is *KFG*?

10. What is *DEH*?

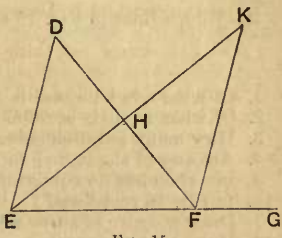


Fig. 15.

XIII.

Draw the following, as nearly exact as you can:—

1. A scalene right-angled triangle.
2. An isosceles obtuse-angled triangle.
3. An equilateral triangle.
4. An isosceles right-angled triangle.
5. A scalene acute-angled triangle.
6. Can you make a triangle with two obtuse angles? Show by a figure.

7. Try if you can make a right-angled equilateral triangle?

8. Can you make an acute-angled isosceles triangle? Show by a figure.

9. Draw a square, as nearly exact as you can.

10. Draw a rhomboid, as nearly exact as you can.

XIV.

1. Draw any triangle, and then make another with its sides the same length as the first, as accurately as you can.

2. Draw six different trapeziums.

3. Of what shape are the diamonds in a pack of cards?

4. How many sorts of figures are there in Fig. 16? Name them all.

5. What is the diameter of the figure *ABCD*? What other name may be given to it, besides "diameter"?

6. There are six pairs of vertically opposite angles in the figure. Find them, and write them down in pairs.

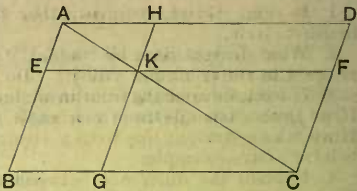


Fig. 16.

7. What are the exterior angles of the triangle *FKC*?

8. Is the figure *KGCF* rectangular?

9. Is *HKF* an angle or a triangle? Of what kind is it? What is *AEK*?

10. Which lines in Fig. 16 appear to be parallel?

XV.

1. Of what quadrilateral is *KC* the diameter?

2. Of what triangle is *BGK* the exterior angle? Give your reasons.

3. How many parallelograms are there in Fig. 16?

4. Are any of the figures in Fig. 16 trapeziums?

5. Is a rhombus an equilateral figure? Is a rhomboid? Why?

6. Is Fig. 16 rectilineal?

7. Draw a figure of three sides, which is not rectilineal.

8. How many diagonals has a square?

9. If two straight lines be drawn so that they would never meet, however far they were produced, are they necessarily parallel? Why?

10. Draw a straight line and produce it till it is twice as long as at first.

XVI.

1. Draw a figure with seven sides. From any point inside the figure draw straight lines to all the angular points. How many triangles does this make? Would the same thing be the case, no matter how many sides the figure had?

2. Draw two parallel straight lines.

3. What is a parallelogram?
4. Is a triangle a parallelogram?
5. What do you mean by the area of a figure?
6. Mention any pairs of parallel lines that you can see round you.
7. If you joined the ends of four equal straight rods together so as to form a quadrilateral figure, of what kind would it be?
8. What do you mean by "coincide"?
9. What is a postulate? an axiom?
10. What is the smallest number of straight lines that can enclose a figure?

XVII.

1. In Fig. 17 which line is the broadest—AB or BC?
2. What do you mean by "equal"? Is the point A *equal* to the point B? Why?
3. If we take an angle from an angle, what is left? If we take a triangle from a triangle, what is left? If we take a straight line from a straight line, what is left? Give examples.
4. Explain the difference between a circle and a ring.

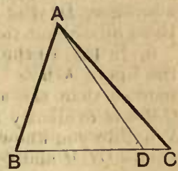


Fig. 17.

5. Can you draw two lines to cut in two points?
6. Which is the greater—BD or BC? Which axiom tells you this?
7. Draw an angle. Now make each of the lines twice as long. Is the angle twice as large as before? State your reasons.
8. In the triangle ADC which angle does AD subtend?
9. What are the names of the angles of the triangle ABC? Is ADC an angle of this triangle?
10. Of what kind is the triangle ADC?

XVIII.

1. In Fig. 18 the lines AB, AC, DE, DF are all the same size. Are the angles BAC, EDF therefore equal?
2. Which is the largest point in the above figure?
3. Try if you can draw a triangle with two right angles.
4. How many acute angles are there in every triangle?

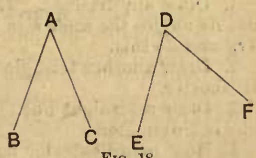


Fig. 18.

5. Is ABC a figure? Why? (Fig. 18.)
6. In Fig. 17 of what triangle is ADB the exterior angle?
7. What instruments are you supposed to have for Euclid? How are they *not* to be used?
8. Is a diameter of a circle a figure?
9. In Fig. 17 which is the larger triangle, ABC or ABD? Why?
10. How does Euclid try if two magnitudes are equal?

XIX.

1. In Greek the word for an angle means "a knee." Can you explain why an angle should be so called?

2. What is the smallest number of straight hurdles with which a sheep could be penned? What axiom tells you this?

3. If a boy A is taller than B, and B is taller than C, what can you assert about the relative height of A and C?

4. In Fig 17 if the angle ADB is greater than the angle ACB, and the angle ABD is greater than the angle ADB, what do you know about the angles ABC and ACB?

5. Two boys A and B are the same height. A has a brother, C, who is taller than he, and B has a brother, D, who is shorter than he. What do you know about the heights of C and D?

6. In Fig. 19 the angle KLM is the same size as the angle KML. The angle QLM is evidently smaller than the angle KLM; while the angle QML is evidently greater than the angle KML. What do you know about the relative size of the angles QLM and QML?

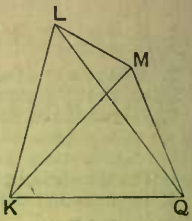


FIG. 19.

7. Make an angle ABC. Now make another angle PQR, so that PQ is the same length as AB, and QR the same length as BC. Does this make the angle PQR equal to the angle ABC? Why?

8. What sort of an angle does your knee generally make when sitting on a form of convenient height? Which would make the larger angle, the knee of a tall boy, or a short one?

9. What is the smallest number of surfaces a body can have?

10. How many surfaces has a hemisphere?

XX.

1. Draw any triangle. It is possible to draw another triangle which has its angles the same size as those of the first, but its sides all longer. Try and do this.

2. Draw another triangle equiangular to the first, but with its sides all shorter.

3. Draw a straight line. Make a quadrilateral of which this straight line is a diameter.

4. How many exterior angles can be drawn to a triangle? Show by a figure.

5. Why is a small angle called "acute"?

6. Is there any connection between a "right" line, and a "right" angle?

7. Which is the *base* of a triangle? Which the vertex?

8. If an angle represents the turn which a man has to make when going from one road to another, show how the angle is named by letters.

9. Can two lines enclose space?

10. Draw two triangles which have one side common to both.

XXI.

1. Draw two triangles which have one angle common to both.
2. What do you mean by "a magnitude"? (as in axiom 8.)
3. Of how many "dimensions" are the following :—a line, a straight line, a brick, a cricket ball, a point, a rectilineal figure, a plane, an angle, a shadow?
4. What is the "perimeter" of a rhomboid, two of whose sides are five inches, and three inches respectively?
5. Show that the outside of one closed surface cannot cut the outside of another closed surface in an odd number of points.
6. What is the meaning of the expression, "Join AB"?
7. How are triangles classed according to their angles?
8. If, when two points are taken on a superficies, the straight line joining them lies wholly in that superficies, is it a plane superficies?
9. Draw two triangles which can coincide, and make them do so.
10. What is the meaning of the word "radius"?

XXII.

1. Draw a straight line, and on it describe a triangle.
2. Is an equilateral triangle isosceles?
3. Where is the centre of a semicircle?
4. What is the diameter of a circle whose radius is 1 inch?
5. The angle DGF is equal to the angle DFG; the angle EFG is greater than DFG; and the angle EGF is less than DGF; what do you know about the angles EFG and EGF?
6. In making a circle with a pair of compasses, how do you know that it fulfils the condition of Def. 15.
7. What is a "hypoteneuse"?
8. If two points are given, how many straight lines can be drawn from one to the other? Why?
9. Draw two triangles as nearly equal as you can. Cut them out, and see if they will coincide.
10. Draw two angles as nearly equal as you can.

XXIII.

1. What is the meaning of "subtend"?
2. Draw a triangle, and produce one of its sides. What new angle have you made, and by what special name is it known?
3. Can you draw two straight lines to cut in two points?
4. Draw two straight lines of equal lengths. Divide each of them into two equal parts. What does Axiom 7 tell you about these?
5. Draw two circles which cut, and then draw their common chord.
6. Draw a rectangle, and divide it into two right-angled triangles.
7. Draw two triangles which have (i.) a common side, (ii.) a common angle.
8. By what is a segment of a circle contained?
9. In what two ways are triangles classified?
10. Is there any connexion between the words "plain" and "plane"?

TO THE LEARNER.

Never attempt to learn a proposition of Euclid "by heart."

First learn *by heart* the General Enunciation, and get a clear idea of what the whole proposition is about. Try and keep this before you, *and never lose sight of it* all through the proposition. Next, read carefully through the whole proposition to get a *general* idea of the contents.

Then go more slowly over each part, and be very careful to *understand* each line before going on to the next. Remember that missing the meaning of one line will often make the whole obscure. When you think that you thoroughly understand it, try and go through the proposition without the book, and with a figure of your own drawing, referring to the book only when you forget a step, and then *begin again* at the Particular Enunciation.

Never copy the figure from the book; but draw it, bit by bit, as you write down the steps of the construction. Do not make your figures and letters too small. Use letters different to those in the book. Draw your figure as accurately as possible, always using a ruler and compasses.

SYMBOLS AND ABBREVIATIONS.

$=$ *for* "equals," *or* "is equal to" (should not be used for the adjective "equal").

$\therefore$	*for*	therefore.	$\because$	*for*	since, because.
$\angle$	,,	angle.	$\angle ABC$	,,	the angle *ABC*.
rt. $\angle$	,,	right angle.	$\perp^{r}$ to	,,	perpendicular to, *or*, at right angles to.
$\triangle$	,,	triangle.	$\parallel^{l}$	,,	parallel.
$\odot^{le}$	,,	circle.	$\square^{gram}$	,,	parallelogram.
$\bigcirc^{ce}$	,,	circumference.			

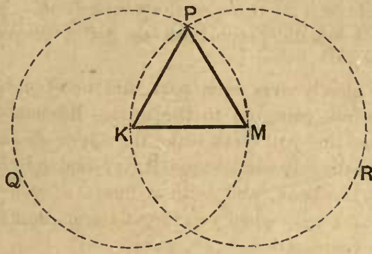
def.	*for*	definition.	ax.	*for*	axiom.
post.	,,	postulate.	prop.	,,	proposition.
hyp.	,,	hypothesis.	const.	,,	construction.
st.	,,	straight.	gr.	,,	greater.
sq.	,,	square.	rect.	,,	rectangle.
tog^{r}.	,,	together.	ext^{r}.	,,	exterior.
cr.	,,	centre.			etc., etc., etc.

PROPOSITION I. PROBLEM.

GENERAL ENUNCIATION—*To describe an equilateral triangle on a given finite straight line.*

PARTICULAR ENUNCIATION—Let KM be the given straight line;

We have to describe an equilateral triangle on KM.



CONSTRUCTION—

1. With centre K and distance KM describe the circle MPQ....................................Post. 3.
2. With centre M and distance MK describe the circle KPR....................................Post. 3.
3. From the point P, where the circumferences cut, draw the straight lines PK and PM..........Post. 1.

Now, we have to prove, that the triangle PKM is equilateral.

PROOF—

1. *Because* K is the centre of the circle MPQ;
   Therefore $KP = KM$..........................Def. 15
2. *Because* M is the centre of the circle KPR;
   Therefore $MP = MK$..........................Def. 15.
3. Now, we have shown that KP and MP are both equal to KM;
   Therefore they are equal to each other....................Ax. 1.

WHEREFORE PK, KM, and MP are all equal, and the triangle PKM is equilateral...............Def. 24.

QUOD ERAT FACIENDUM.

(Which was required to be done.)

PROPOSITION II. PROBLEM.

GENERAL ENUNCIATION—*From a given point to draw a straight line equal to a given straight line.*

PARTICULAR ENUNCIATION—Let P be the given point and XY the given straight line;

We have to draw, from P, a straight line of the same length as XY.

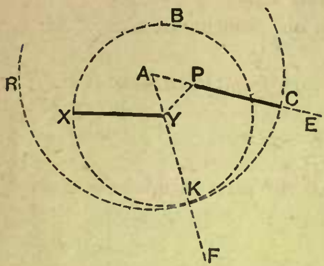


FIG. 1.

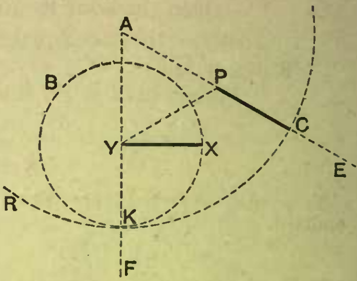


FIG. 2.

CONSTRUCTION—

1. Join P to one end of the line XY (*say* Y).........Post. 1.
2. On this *new* line PY describe an equilateral triangle PAY...I.1.
3. Produce AP, AY (*the two new sides*) to E and F...Post. 2.
4. With centre Y (*the end which was joined to* P) and distance YX (*the given line*), describe the circle XKB....................................Post. 3. Cutting the line AF (*which passes through the centre* Y) at K.
5. With centre A and distance AK (*to the new point*) describe the circle RKC..................Post. 3. Cutting the straight line AE (*the other produced line*) in C.

Now we have to prove that PC is equal to XY.

PROOF—

1. *Because* Y is the centre of the circle XKB;
   Therefore $YX = YK$ ....Def. 15.
2. *Because* A is the centre of the circle RKC;
   Therefore $AK = AC$ ....Def. 15.
3. But $AY = AP$ (*sides of equilateral triangle*);
   Therefore the remainder YK = the remainder PC ...Ax. 3.
4. But we showed above that YX was equal to YK; *Therefore* PC and XY are each of them the same length as YK.

   Therefore they are equal to one another.............Ax. 1.

WHEREFORE

The straight line PC is drawn from P, equal to XY.

Q.E.F.

NOTE.

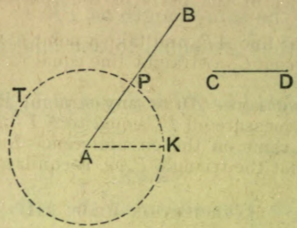
In writing out this Proposition, all the parts in brackets may be omitted.

PROPOSITION III. PROBLEM.

GEN. ENUN.—*From the greater of two given straight lines to cut off a part equal to the less.*

PART. ENUN.—Let AB and CD be the two given straight lines, of which AB is the greater;

We have to cut off a part from AB, of the same length as CD.



CONSTRUCTION—

1. From the point A draw a straight line AK equal to CD..I. 2.
2. With centre A, and at the distance AK, describe the circle KPT cutting AB at P.

Now we have to prove that the part AP, cut off from AB, is equal to CD.

PROOF—

Because A is the centre of the circle KPT;
Therefore $AP = AK$..............................Def. 15.
But $AK = CD$ (it was made equal).......Const.
Therefore $AP = CD$...................................Ax. 1.

WHEREFORE *from the straight line AB, etc.*

Q.E.F.

EXERCISES ON PROP. I.

1. From what other point could we draw straight lines to K and M, instead of P?
2. Write out the Proposition, doing it in that way.

3. If we join P to K and M, and also the other point to K and M, prove that the figure we get is a rhombus.

4. How much of the circumferences of the two circles is absolutely necessary in order to construct the triangle?

Exercises on Prop. II.

1. There are, in all, eight ways of constructing the figure for this proposition, of which two are given above. For the point may be joined to either end of the line (two ways), then the equilateral triangle may be described on either side of the line (four ways), and its sides may be produced in either direction (making eight ways). Try and draw these, taking especial notice of the parts of the construction which are in brackets.

2. Draw a straight line AB, and take a point C in it. Construct the figure for drawing from C, a straight line equal to AB, and prove your result.

3. Why cannot we say—"Draw any straight line PE from P, and with the compasses measure off PC equal to XY"?

4. If the point A came on the circumference KBX, where would P lie? (Remember that the triangle PAY is equilateral.)

Exercises on Prop. III.

1. Produce the smaller of two straight lines, so that it may be equal to the greater.

2. In I. 3 why cannot we measure CD with a pair of compasses, and mark off the length on AB?

3. Draw two straight lines, and make an isosceles triangle on the smaller one as base, and with its sides equal to the larger one.

4. From AB cut off a part equal to twice CD.

5. Draw the figure of this Proposition with all the construction lines required for Props. I. and II., and prove your result.

PROPOSITION IV. Theorem.

Gen. Enun.—*IF two triangles have*

(What is given.) *Two sides of the one triangle equal to two sides of the other triangle, each to each,*
And have also the angles contained by these sides equal,

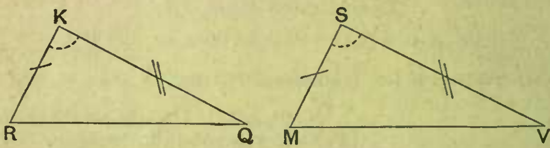
THEN

(What is to be proved.) *They* shall *have their bases (or third sides) equal,*
And the two areas shall *be equal,*
And the other angles shall *be equal, each to each* (viz., those to which the equal sides are opposite).

Part. Enun.—IF the two triangles *KRQ* and *SMV* have

(What is given.)

1. The side *RK*	equal to	the side *MS*
2. and the side *KQ*		and the side *SV*,
3. and their angle *RKQ*		and their angle *MSV*.



THEN we have to prove

(What is to be proved.)

1. **that the base** *RQ*	**is equal to**	**the base** *MV*,
2. **that the area** *RKQ*	,,	**the area** *MSV*,
3. **that the angle** *KRQ*	,,	**the angle** *SMV*,
4. **that the angle** *KQR*	,,	**the angle** *SVM*.

Construction—

Put the triangle *KRQ* on the triangle *SMV*,
so that the point *K* lies on the point *S*,
and the line *KR lies along* the line *SM*.

Proof—

1. *Because* *KR* is *equal* to *SM*....................Given.
Therefore *KR coincides* with *SM*;
and *therefore* the point *R* will lie on the point *M*.

2. *Because KR lies along SM* (it was put there)........Const.
and the angle RKQ is *equal* to the angle MSV.Given.
Therefore KQ will *lie along SV*.

3. *Because* KQ *lies along* SV..............Just proved.
and KQ is *equal* to SV.....................Given.
Therefore KQ *coincides with* SV;
and *therefore* the point Q lies on the point V.

But also, the point R lies on the point M...Proved in part 1.

4. *Because* the two ends of the line RQ lie on the two ends of the line MV,
Therefore the line RQ *coincides with* the line MV
(For, if it did not coincide, these two straight lines would enclose a space, which is impossible)....Ax. 10.

THEREFORE we have now proved that

1. The base RQ coincides with, and is equal to the base MV Ax. 8
2. The area RKQ „ „ „ the area MSV „
3. The angle KRQ „ „ „ the angle SMV „
4. The angle KQR „ „ „ the angle SVM „

WHEREFORE *if two triangles have, etc.*

QUOD ERAT DEMONSTRANDUM.
(Which was required to be proved.)

NOTES.

In the proof of this proposition, we first show that the left-hand sides of the triangle coincide; then that the angles between the sides coincide; and then that the right-hand sides coincide.

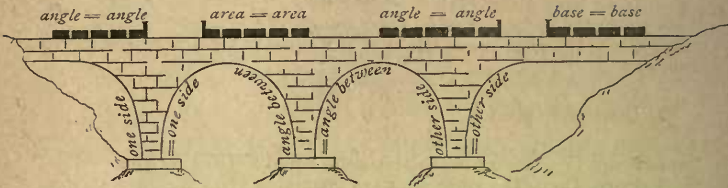
A triangle has seven parts, viz., three sides, three angles, and an area; and this Proposition shows that when a certain *three* of these parts are equal, in two triangles, then the other *four* must be equal also.

Having proved this Proposition, we shall now have to apply it to prove other Propositions; and, in doing so, it will, of course, not be necessary to prove it over again each time. The following illustration will perhaps make this clearer:—

When the Forth Bridge was completed, some very heavily laden trains were sent over it, of greater weight than it will be necessary ever to send over again. As the bridge stood this test, the ordinary traffic can now be sent over it without the need for testing it each time.

NOTES.

Let us take a bridge to represent the Proposition.



Then the three pillars represent three *facts*.

No. 1. The *fact* that

One side in the first triangle	=	One side in the second triangle.

No. 3. The *fact* that

Another side in the first triangle	=	Another side in the second triangle.

No. 2 (*between* 1 and 3). The *fact* that

The angle *between* the sides in the first triangle	=	The angle *between* the sides in the second triangle.

Now these pillars must rest on firm *foundations*; *i.e.*, we must have good *reasons* for stating the three facts.

If, then, the three pillars and their foundations are secure, we can send four trains over the bridge, because it has been tested for these four in proving Prop. IV.

They are:—

1st train—The *fact* that the bases are equal.
2nd train—The *fact* that the areas are equal.
3rd train—The *fact* that the second angle = the second angle.
4th train—The *fact* that the third angle = the third angle.

We may not always have to send all four over at once; but we know that as our bridge has stood the test for four, we can trust it to bear any of them, provided that the three pillars are there, and their foundations firm.

It is easy to see that if one of the pillars is wanting our train will collapse at that point; or the same thing will occur if the pillar rests on an insecure foundation.

Remember that the first pillar is not one side of one triangle, or one side of each triangle, but *a fact*,* and so for the other pillars.

Again, the trains represent, *not* bases, or areas, or angles, but *facts* about these bases, areas, and angles.

* What fact?

EXERCISES.

1. If we have given $\begin{cases} KR = SM, \\ RQ = MV, \\ \text{angle } RKQ = \text{angle } MSV, \end{cases}$

does Prop. IV. then prove the triangles equal? Give a reason for your answer.

2. If we have given $\begin{cases} KQ = SV, \\ RQ = MV, \\ \text{angle } KQR = SVM, \end{cases}$

does Prop. IV. then prove the triangles equal? Give reasons.

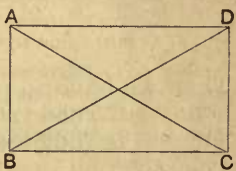
3. Prove Prop. IV. when we have given

$$\left.\begin{array}{r} \text{the side } KR \\ \text{and the side } RQ \\ \text{and their angle } KRQ \end{array}\right\} \text{equal to} \left\{\begin{array}{l} \text{the side } SM, \\ \text{and the side } MV, \\ \text{and their angle } SMV. \end{array}\right.$$

4. ABC is an equilateral triangle, and the angle BAC is divided into two equal parts by a straight line AD, which meets BC in the point D. Draw the figure. and say what you know about the triangles ABD and ACD, and what Prop. IV. proves about them.

5. If two triangles have two sides of the one equal to two sides of the other, each to each; are the bases equal? Why?

6. Prove by Prop. IV. that if $ABCD$ be an oblong, then $AC = BD$. (Take the triangles ABC and DCB.) *N.B.*—In an oblong the opposite sides are equal.

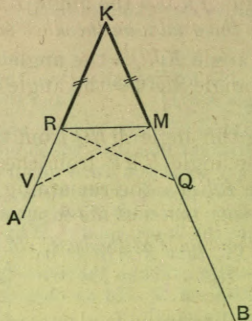


7. In Prop. IV. if the point R lay on M, and Q on V, and yet the straight line RQ did not coincide with MV, what would they look like? Show by a figure. What do you see from the figure?

PROPOSITION V. THEOREM.

GEN. ENUN.—*The angles at the base of an isosceles triangle are*
(To be proved.) *equal to each other; and if the equal sides be produced, the angles on the other sides of the base are also equal.*

PART. ENUN.—*Let KRM* be an isosceles triangle,
having the side *KR* equal to the side *KM*,
(Given.) and their equal sides produced to *A* and *B* respectively.



Then we have to prove
(To be proved.) **1. That the angle** *KRM* = **the angle** *KMR*.
2. That the angle *ARM* = **the angle** *BMR*.

CONSTRUCTION—
1. In *RA* take any point *V*
2. From *KB* cut off *KQ* equal to *KV*......................I. 3.
3. Join *RQ, VM*.

PROOF—
1. In the two triangles *KRQ* and *KMV*
Because we have

the sides *RK*	equal to	the sides *MK*	Given.
and *KQ*		and *KV*	Const.
and the angle *RKQ*		and the angle *MKV*	The same.

Therefore
the base *RQ* = the base *MV*
the angle *KRQ* = the angle *KMV* ....I. 4.
the angle *KQR* = the angle *KVM*
(*i.e.*, the angle *MQR* = the angle *RVM*)...Note on Def. 9.

2. In the two triangles RVM and RQM

Because we have

$$\left.\begin{array}{r}\text{the sides } RV^* \\ \text{and } VM\dagger \\ \text{and the angle } RVM\end{array}\right\} \begin{array}{c}\text{equal}\\ \text{to}\end{array} \left\{\begin{array}{ll}\text{the sides } MQ & \text{Const.} \\ \text{and } QR & \text{Proved in (1).} \\ \text{and the angle } MQR & \text{Proved in (1).}\end{array}\right.$$

Therefore

$$\left.\begin{array}{r}\text{the angle } RMV = \text{the angle } MRQ \\ \text{and the angle } VRM = \text{the angle } QMR\end{array}\right\} \text{..........I. 4.}$$

(*i.e.*, the angle ARM = the angle BMR)...Note on Def. 9.

And these are the angles on the other side of the base.

3. *Because* the angle KRQ = the angle KMV...Proved in (1).

and the angle MRQ = the angle RMV...Proved in (2).

Therefore

(taking the angle MRQ from the angle KRQ

and the angle RMV from the angle KMV)

the remaining angle KRM = the remaining angle KMR...Ax. 3.

And these are the angles at the base.

WHEREFORE, *the angles at the base, etc.*

Q.E.D.

COROLLARY—

Every equilateral triangle is also equiangular.

NOTES.

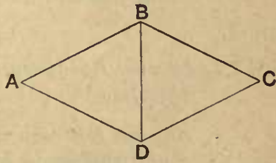
Compare carefully Part 1 of the above proof with the Particular Enunciation of Prop. IV.

Notice that Prop. IV. is first applied to the two *large* triangles KRQ and KMV; and then in Part 2 to the *small* triangles RVM and MQR.

A Corollary is a sort of minor conclusion, the truth of which easily follows from the main proposition.

EXERCISES.

1. How far must the equal sides be produced?
2. Must they both be produced to the same length?
3. Prove the corollary.
4. $ABCD$ is a rhombus, with BD joined. Prove that the angle ABC = the angle ADC.



5. Write out the proof of Prop. V. when the sides KR and RM are given equal, and produced.
6. Is $AV = BQ$? Why?

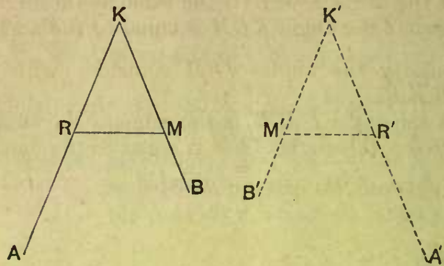
* Because $KR = KM$ and $KV = KQ$. (Ax. 3.)

† Notice that the side RM is *not* mentioned.

PROPOSITION V. THEOREM. (Second Proof.)

GEN. ENUN.—*The angles at the base of an isosceles triangle are* (To be proved.) *equal to each other; and if the equal sides be produced, the angles on the other side of the base are also equal.*

PART. ENUN.—*Let KRM* be an isosceles triangle,
having the side *KR* equal to the side *KM*,
(Given.) and these equal sides produced to *A* and *B* respectively.



Then we have to prove

(To be proved.)
1. **That the angle** *KRM* = **the angle** *KMR*.
2. **That the angle** *ARM* = **the angle** *BMR*.

CONSTRUCTION—

Suppose the figure to be taken up, turned over, and laid down again.
Then it will fall as in the dotted figure *K′M′R′*.
Put the first figure on the dotted one
So that the point *K* comes on the point *K′*
and the line *KRA* on the line *K′M′B′*.

PROOF—

1. *Because KR* is equal to *K′M′* ....Given.
Therefore the point *R* coincides with the point *M′*.

2. *Because KR* coincides with *K′M′*..............Just proved.
and the angle *RKM* is equal to the angle *M′K′R′*.
Therefore KMB lies along *K′R′A′*.

3. *Because* KM lies along $K'R'$ ....Just proved.
and KM is equal to $K'R'$ .... Given.
Therefore M coincides with R'.
But also R coincides with M'................Proved above.
Therefore RM coincides with $M'R'$.

4. *Therefore* all the lines of the first figure *lie along* the lines of the second figure.*
Therefore the angle KRM coincides with the angle $K'M'R'$.
But the angle $K'M'R'$ is the same as the angle KMR.
Therefore the angle KRM is equal to the angle KMR.

Similarly the angle ARM coincides with the angle $B'M'R'$.
But the angle $B'M'R'$ is the same as the angle BMR.
Therefore the angle ARM is equal to the angle BMR.

WHEREFORE *the angle at the base, etc.*

Q.E.D.

COROLLARY—

Every equilateral triangle is also equiangular.

NOTE.

A Corollary is a sort of minor conclusion, the truth of which easily follows from the main proposition.

EXERCISES.

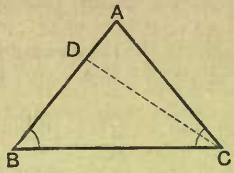
1. How far must the equal sides be produced?
2. Must they both be produced to the same length?
3. Prove the corollary.
4. $ABCD$ is a rhombus, with BD joined. Prove that the angle ABC=the angle ADC.
5. Write out the proof of Prop. V. when the sides KR and RM are given equal and produced.
6. Is $AR=BM$? Why?

* Notice that the points A and B will not necessarily coincide with the points B' and A', and so we cannot say that all the lines *coincide*.

PROPOSITION VI. THEOREM.

GEN. ENUN.—*IF two angles of a triangle are equal to each other, THEN the sides which subtend (i.e., are opposite to) the equal angles, are also equal to each other.*

PART. ENUN.—Let ABC be a triangle, having the angle ABC equal to the angle ACB. (Given.)



(To be proved.) We have to prove that the side AB = the side AC.

HYPOTHESIS—

Suppose that AB is *greater* than AC.

CONSTRUCTION—

From BA *cut off a part* BD *equal to* AC..................I. 3.

Join CD.

PROOF—

If in the triangles DBC and ACB

we have the sides DB and BC and the angle DBC = the sides AC.....Hyp. and Cons. and CB................The same. and the angle ACB.....Given.

Therefore the *area* DBC = the *area* ACB...............I. 4.

Which is evidently absurd.

Therefore it is absurd to suppose AB is greater than AC.
Similarly we could show that AB cannot be less than AC.
Therefore AB must be equal to AC.

WHEREFORE *if two angles of a triangle, etc.*

Q.E.D.

COROLLARY—

Every equiangular triangle is also equilateral.

NOTES.

This Proposition is called the *converse* of Prop. V.

Prop. V. proves { If the sides are equal / Then the angles are equal.

Prop. VI. proves { If the angles are equal / Then the sides are equal.

This method of proof is called "Reductio ad absurdum." In order to prove a fact we *suppose* that just the opposite may be possible. Then we show that this results in an absurdity, and therefore our supposition cannot be true, and therefore the original proposition is true. This is sometimes called an *Indirect* method of proof.

Avoid the mistake of saying, in the construction, "In AB take a point D." We could not then even *suppose* BD to be equal to AC.

The word "hypothesis" means "supposition." The word is generally used in Euclid for a correct or an incorrect supposition. For the present, however, we shall always use it as denoting a *mere* supposition, which has to be shown to be incorrect.

EXERCISES.

1. Prove the Proposition by supposing AB to be less than AC.
2. Prove the Corollary.
3. When we have shown that AB is not greater than AC, do we know that AB is equal to AC?
4. Why is the area DBC not equal to the area ACB?

PROPOSITION VII. Theorem.

Gen. Enun.—*On the same base, and on the same side of it, there cannot be two triangles having the sides terminated at one end of the base equal to each other,* IF *the sides terminated at the other end of the base be also equal to each other.*

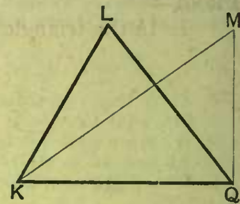
Hypothesis—Suppose it possible that on the same base KQ, and on the same side of it, there *can be* two triangles, KLQ and KMQ, which have

(1.) the side KL = the side KM (both terminated at K),

and also (2.) the side QL = the side QM (both terminated at Q), *at the same time.*

This Proposition has three cases.

Case I.—Where the vertex of *each* triangle falls outside the other triangle.

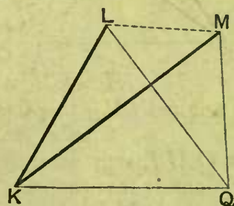


Construction—

Join the vertices L and M.

Proof—

1. In the triangle KLM,



If $KL = KM$................Hypothesis.

Therefore the angle KLM = the angle KML...........I. 5.

But the angle QML is gr. than the angle KML } Ax. 9.
And the angle QLM is less than the angle KLM }

Therefore the angle QML is *much* gr. than the angle QLM.

2. In the triangle QLM,

If $QM = QL$..................Hypothesis.

Therefore the angle QML = the angle QLM...........I. 5.

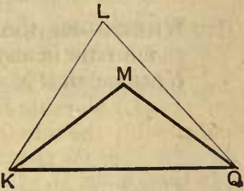
But we have just proved (in 1) that the angle QML is much greater than the angle QLM;

Therefore, *if the Hypothesis be true*, the angle *QML* is both equal to, and much greater than, the angle *QLM*,

Which is impossible;

THEREFORE the supposition made above is *not* true for the first case.

CASE II.—Where the vertex (*M*) of the *one* triangle falls *inside* the other triangle.



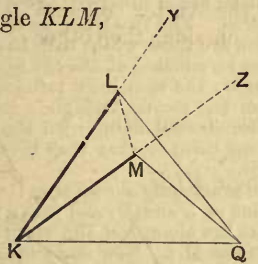
CONSTRUCTION—

Join the vertices *L* and *M*.

Produce *KL* and *KM* to *Y* and *Z* respectively.

PROOF—

1. In the triangle *KLM*,



If *KL* = *KM*................Hypothesis.

Therefore the angle *YLM* = the angle *ZML* (*on the other side of the base*)......I. 5.

But the angle *QML* is gr. than the angle *ZML* }
And the angle *QLM* is less than the angle *YLM* } Ax. 9.

Therefore the angle *QML* is *much* gr. than the angle *QLM*.

2. In the triangle *QLM*,

If *QM* = *QL*..................Hypothesis.

Therefore the angle *QML* = the angle *QLM*...........I. 5.

But we have just proved (in 1) that the angle *QML* is much greater than the angle *QLM*;

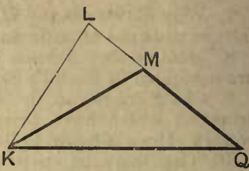
Therefore, *if the Hypothesis be true*, the angle *QML* is both equal to, and much greater than, the angle *QLM*,

Which is impossible;

THEREFORE the supposition made above is *not* true for the second case.

CASE III.—Where the vertex (M) of one triangle falls *on a side* of the other triangle,

In this case it is *evident* that LQ cannot be equal to MQ...Ax. 9.



WHEREFORE the supposition is not true in any case, and

Upon the same base, and on the same side of it, etc.

Q.E.D.

NOTES.

In Cases I. and II. of this Proposition the given triangles are those shown by the thick and thin lines respectively of the upper figure. But in the proof we make use of other triangles, and the figure is given again, somewhat altered. The two triangles on which the learner has now to fix his attention are those which both have the dotted line LM as base, the thick lines terminated at the *left hand* end of KQ being the sides of the first triangle; the thin lines terminated at the *right hand* end of KQ, the sides of the second triangle. Both these triangles are *supposed* to be isosceles, and beginning with the base angles of the *left hand* triangle we change them so that they become the base angles of the *right hand* triangle.

This change may be practically shown thus:—

Put two drawing pins at L and M, and pass a fine string round them, so that its middle part lies along LM and the two ends along LK and MK (or YL, ZM in Case II.). Now move the strings so that the one which was on KM (or ZM) shall lie on QM, and the other on QL. In doing this we increase the angle that the string formed at M, and decrease the one that it formed at L.

These two new angles are those of the *right hand* triangle, and we show, as in the Proposition, that the supposition we made leads us to the conclusion that they are both equal and unequal to each other, which is evidently absurd.

This Proposition is only required in the proof of Prop. VIII.; and it becomes unnecessary if the Second Proof of that Proposition is used.

EXERCISES.

1. Can we have the side KL equal to the side KM?
2. What is the method of proof called which is used in this Proposition? Explain the meaning of your answer.
3. What is the difference between the proof of Case I. and that of Case II.?
4. In Case III. are KL and KM equal? Give your reason.
5. In Case I. can we have $QL = QM$? Show by a figure.
6. What is meant by "the vertex of a triangle"?

7. Why do we not say in Case I. "Where the vertex of one triangle falls outside the other triangle"?

8. Mention any Axioms which are assumed in this Proposition, and are not given in Euclid's list.

9. Can there be, on the same base and on the same side of it, two triangles which have their sides terminated at one end of the base equal? Draw a figure to illustrate your answer.

10. Can there be on the same base and on the same side of it, two triangles which have the sides of the first triangle equal to the sides of the second, each to each? Show by a figure.

11. Can there be on the same base two triangles which have the sides terminated at one end of the base equal, if those terminated at the other end of the base are also equal? Show by a figure.

12. Prove that on the same base, and on the same side of it, there can be only one equilateral triangle.

13. If a rhombus $ABCD$ be folded along its diagonal AC, how will the points B and D lie with regard to each other?

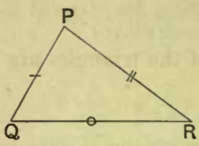
PROPOSITION VIII. THEOREM.

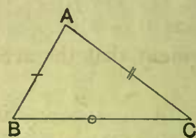
GEN. ENUN.—*IF two triangles have the three sides of the one equal* (Given.) *to the three sides of the other, each to each,*

(To be proved.) *THEN the angle which is contained by any two sides of the one triangle* } *is equal to* { *the angle contained by the two corresponding sides of the other triangle.*

PART. ENUN.—LET the two triangles PQR and ABC have

(Given.) The sides PQ and QR and RP } equal to { the sides AB and BC and CA, each to each.





THEN we have to prove that

(To be proved.)
1. **The angle** PQR = **the angle** ABC,
2. **The angle** QRP = **the angle** BCA,

and 3. **The angle** RPQ = **the angle** CAB.

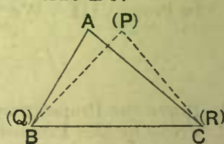
CONSTRUCTION—

Put the triangle PQR on the triangle ABC so that the point Q lies on the point B* and the side QR lies along the side BC.

PROOF—

Because $QR = BC$........................Given.

Therefore the point R will lie on the point C and the side QR *coincide* with the side BC.



Now, if the other two sides PQ and PR do not coincide with the sides AB and AC, they will take up some other position such as $(P)(Q)(R)$

But in this case we should have on the same base BC and on the same side of it two triangles ABC and $(P)(Q)(R)$, which have $AB = (P)(Q)$ and $AC = (P)(R)$, which is impossible.....I. 7.

* *Not* P on A as in Prop. IV.

Therefore the two sides PQ and PR *must* coincide with the two sides AB and AC.

And since all the three sides of the triangle ABC coincide with all three sides of the triangle PQR,

Therefore the angles also coincide, and so are equal.

THEREFORE we have proved as required that

1. The angle PQR = the angle ABC.
2. The angle QRP = the angle BCA.

and 3. The angle RPQ = the angle CAB.

WHEREFORE,

If two triangles have, etc.

Q.E.D.

COROLLARY—

It is evident that the areas of the triangles are also equal.

NOTES.

The enunciation of this proposition is sometimes given as follows :—

If two triangles have two sides of the one equal to two sides of the other, each to each, and have likewise their bases equal, the angle which is contained by the two sides of the one shall be equal to the angle which is contained by the two sides, equal to them, of the other.

Compare this enunciation with that in Prop. IV.

In *both* cases we have two triangles having two sides of the one equal to two sides of the other, but

In Prop. IV. { We have, in addition, the contained angles equal, *To prove* the bases equal.

In Prop. VIII. { We have, in addition, the bases equal, *To prove* the contained angles equal.

So, in Prop. IV. { We begin by putting the vertex of the *given angle* on the other vertex.

In Prop. VIII. { We begin by putting the *given base* on the other base.

EXERCISES.

1. Prove the Proposition, beginning by putting the side PR on AC.

2. Prove by this Proposition that the diagonal of a rhomboid divides it into two parts of equal area.

3. ABC is an equilateral triangle, and D the middle point of BC. Prove by Prop. VIII. that the straight line AD divides the angle BAC into two equal angles.

4. Show what other positions the dotted triangle on page 38 might be supposed to take.

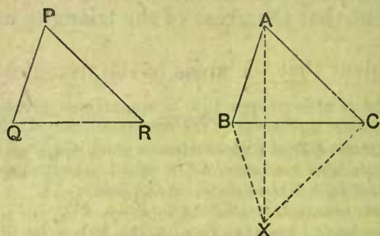
PROPOSITION VIII. THEOREM. (SECOND PROOF.)

GEN. ENUN.—*IF two triangles have the three sides of the one equal*
(Given.) *to the three sides of the other, each to each,*

(To be proved.) *THEN the angle which is contained by any two sides of the one triangle* } *is equal to* { *the angle contained by the two corresponding sides of the other triangle.*

PART. ENUN.—LET the two triangles PQR and ABC have

(Given.) The sides PQ and QR and RP } equal to { the sides AB and BC and CA, each to each.



Then we have to prove that

(To be proved.)
1. **The angle** PQR = **the angle** ABC,
2. **The angle** QRP = **the angle** BCA,

and 3. **The angle** RPQ = **the angle** CAB.

CONSTRUCTION.—

Put the triangle PQR so that the point Q lies on the point B, and the straight line QR on the straight line BC;
But, so that the vertex P shall fall on the opposite side of BC, from the vertex A.
It will then take the position BXC (since $QR = BC$).
Join AX.

PROOF—

1. *Because* $BX = BA$ .... Given.
Therefore the angle BXA = the angle BAX.......... I. 5.

Similarly, the angle CXA = the angle CAX.

2. *Therefore*, adding the angle BXA to the angle CXA, and the angle BAX to the angle CAX,
The whole angle BXC = the whole angle BAC....Ax. 2.
i.e., the angle QPR = the angle BAC.

3. In the triangles PQR and ABC,
Because we have

$$\left.\begin{array}{r}\text{the sides } QP\\ \text{and } PR\\ \text{and the angle } QPR\end{array}\right\}\begin{array}{c}\text{equal}\\ \text{to}\end{array}\left\{\begin{array}{l}\text{the sides } BA \text{.............Given.}\\ \text{and } AC \text{.............Given.}\\ \text{and the angle } BAC \text{ Just proved.}\end{array}\right.$$

$$\textit{Therefore}\quad\left.\begin{array}{r}\text{the angle } PQR = \text{the angle } ABC\\ \text{and the angle } QRP = \text{the angle } BCA\end{array}\right\}\text{.......I. 4.}$$

WHEREFORE, *if two triangles have, etc.*

Q.E.D.

COROLLARY—
It is evident that the areas of the triangles are also equal.

NOTES.

The enunciation of this Proposition is sometimes given as follows:—
If two triangles have two sides of the one equal to two sides of the other, each to each, and have likewise their bases equal, the angle which is contained by the two sides of the one shall be equal to the angle which is contained by the two sides, equal to them, of the other.

Compare this enunciation with that in Prop. IV.

In *both* cases we have two triangles having two sides of the one equal to two sides of the other, but

In Prop. IV. { We have, in addition, the contained angles equal,
To prove the bases equal.

In Prop. VIII. { We have, in addition, the bases equal,
To prove the contained angles equal.

So, in Prop. IV. { We begin by putting the vertex of the *given angle* on the other vertex.

In Prop. VIII. { We begin by putting the *given base* on the other base.

EXERCISES.

1. Prove the Proposition, beginning by putting the side PR on AC.

2. Prove by this Proposition that the diagonal of a rhomboid divides it into two parts of equal area.

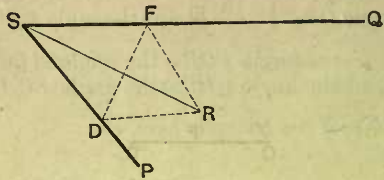
3. ABC is an equilateral triangle, and D the middle point of BC. Prove by Prop. VIII. that the straight line AD divides the angle BAC into two equal angles.

PROPOSITION IX. PROBLEM.

GEN. ENUN.—*To bisect a given rectilineal angle* (that is, to divide it into two *equal* parts).

PART. ENUN.—Let *PSQ* be the given rectilineal angle.

It is required to bisect it.



CONSTRUCTION—

1. In *SP* take any point *D*.
2. From *SQ* cut off a part *SF* equal to *SD*, and join *DF*...I. 3.
3. On *DF*, on the side remote from *S*, describe an equil[1] triangle *FDR*,..............................I. 1.
4. Join *SR*.

Now we have to prove that *SR* bisects the angle *PSQ*.

PROOF.—*Because* in the triangles *DSR* and *FSR*

we have $\left.\begin{array}{r} DS \\ \text{and } SR \\ \text{and } RD \end{array}\right\} = \left\{\begin{array}{l} FS \text{................Const. (2.)} \\ \text{and } SR \text{............Common.} \\ \text{and } RF \text{.........Const. (3.)} \end{array}\right.$

Therefore the angle *DSR* = the angle *FSR*..........I. 8.
i.e., the angle *PSR* = the angle *QSR*, Note on Def. 9.

WHEREFORE, *the angle PSQ is bisected by the straight line SR.*
Q.E.F.

EXERCISES.

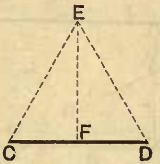
1. Where ought we to suppose the straight line (without breadth), represented by the mark *SR*, to lie?
2. Why do we say in the Construction (part 3) "on the side remote from *S*? Would not the other side do as well?
3. Divide a given angle into four equal parts.

PROPOSITION X. PROBLEM.

GEN. ENUN.—*To bisect a given finite straight line* (*i.e.*, to divide it into two *equal* parts).

PART. ENUN.—Let *CD* be the given finite st. line.

It is required to bisect it.



CONSTRUCTION—

1. On *CD* describe an equilat^{l} triangle *CED*.............I. 1.
2. *Bisect the angle CED*, by *EF*, cutting *CD* in *F*,.......I. 9.

Then *shall CD* **be bisected at** *F*.

PROOF—

Because in the triangles *CEF* and *DEF*,

$$\text{we have}\quad \left.\begin{array}{r} CE \\ \text{and } EF \\ \text{and their angle } CEF \end{array}\right\} = \left\{\begin{array}{ll} DE \text{..........................} & \text{Const. (1).} \\ \text{and } EF \text{..................} & \text{Common.} \\ \text{and their angle } DEF \text{...} & \text{Const. (2.)} \end{array}\right.$$

Therefore the base *CF* = the base *DF*.....................I. 4.

WHEREFORE, *the given st. line CD is bisected at the pt. F.*

Q.E.F.

EXERCISES.

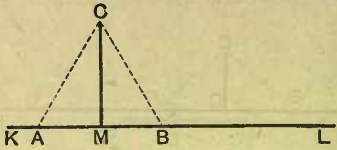
1. In Part 2 of the Construction, the mistake is sometimes made of saying, "In *CD* take a point *F*, and join *EF*." Why is this wrong?
2. Why is the straight line said to be "finite"?
3. Divide a given straight line into eight equal parts.
4. Will this Construction enable us to divide a straight line into any number of equal parts?
5. Why do you learn how to bisect an angle, before learning how to bisect a straight line?

PROPOSITION XI. PROBLEM.

GEN. ENUN.—*To draw a straight line at right angles to a given straight line, from a given point in the same.*

PART. ENUN.—Let KL be the given st. line, and M the given point in it.

It is required to draw from M a st. line at right angles to KL.



CONSTRUCTION—

1. In MK take any point A.
2. From ML cut off MB equal to MA..................I. 3.
3. On AB describe an equil' triangle ABC..............I. 1.
4. Join CM.

Then shall CM be at right angles to KL.

PROOF—

Because in the triangles ACM, BCM,

we have $\begin{cases} AC = BC \text{..........................Const. (3).} \\ CM = CM \text{..........................Common.} \\ MA = MB \text{..........................Const. (2).} \end{cases}$

Therefore the angle AMC = the angle BMC..........I. 8.

But when one straight line, etc., ...Def. 10.
Therefore each of the angles CMA, CMB is a right angle.

WHEREFORE *CM has been drawn at right angles to KL.*

Q.E.F.

EXERCISES.

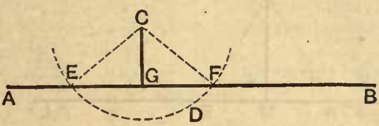
1. Why cannot we say in Part 2 of the Construction, "From LM cut off LB equal to KA"?
2. Is it necessary that the triangle ABC should be equilateral?
3. Draw a straight line AB. From the point A draw a straight line at right angles to AB.

PROPOSITION XII. Problem.

Gen. Enun.—*To draw a straight line perpendicular to a given straight line of unlimited length, from a given point* without *it.*

Part. Enun.—Let AB be the given st. line of unlimited length, and C the given pt. without it.

It is required to draw from C a st. line perpendicular to AB.



Construction—

1. Take any pt. D, on the other side of AB.
2. With cr. C and distance CD, describe the ⊙le EDF, cutting AB in the pts. E and F.
3. *Bisect EF in G*, and join CG, CE, CF.................I. 10.

The straight line CG shall be perpendicular to AB.

Proof—

Because in the triangles ECG and FCG,

we have $\begin{cases} EC = FC \text{..........................Const. (radii).} \\ CG = CG \text{...............................Common.} \\ GE = GF \text{.............................Const. (3).} \end{cases}$

Therefore the angle EGC = the angle FGC............I. 8.
But when one st. line, etc......................Def. 10.

Wherefore, *from the pt. C, CG has been drawn perpendicular to AB.*

Q.E.F.

Exercises.

1. Why is the straight line said to be "of unlimited length"?
2. Would it do to take D on the same side of AB as C is?
3. How much of the circle EDF is it absolutely necessary to draw, in practice?
4. What difference would it make in the Proof if, instead of bisecting EF in G (Const. 3), we bisected the angle ECF by CG meeting AB in G? Write out the proof for this method.

PROPOSITION XIII. THEOREM.

GEN. ENUN.—*The angles which one straight line makes with another straight line on the same side of it either* are *two right angles, or* are together equal to *two right angles.*

PART. ENUN.—Let the straight line XY make with the straight line AB on one side of it the angles XYA, XYB.

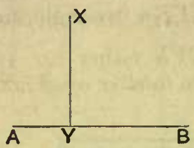


FIG. 1.

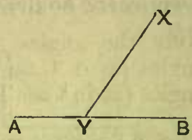


FIG. 2.

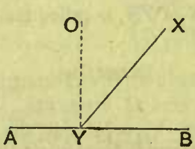
We have to prove that these either *are* two right angles, or are *together equal to* two right angles.

CASE I.—Where the angle XYA is equal to the angle XYB (Fig. 1).

PROOF—

By Definition 10 these *are* two right angles.

CASE II.—Where the angle XYA is not equal to the angle XYB (Fig. 2).



CONSTRUCTION—

At the pt. Y draw YO at right angles to AB..........I. 11.

PROOF—

Then, the angles OYA, OYB *are* two right angles.

1. The angle OYB = the angles OYX and XYB...(Evident).

Add to each the angle OYA.

Therefore the angles OYA, OYB = the angles OYA, OYX, XYB..................................Ax. 2.

2. The angle XYA = the angles OYA, OYX......(Evident).
Add to each the angle XYB.
Therefore the angles XYA, XYB = the angles OYA, OYX, XYB.................................. Ax 2.

3. *But also* the angles OYA, OYB = the angles OYA, OYX, XYB..............................Proved in (1).
Therefore the angles XYA, XYB = the angles OYA, OYB.....................(which *are* two right angles).
That is, the angles XYA, XYB = two right angles.

WHEREFORE the angles XYA, XYB either *are* two right angles (as in Case I.), or are *together equal to* two right angles (as in Case II.), and,

The angles which one straight line, etc.

Q.E.D.

NOTES.

In the figure for Case II. there are two angles which are divided into parts, AYX and OYB. We first take the angle OYB, of which OY is an arm, and then AYX, of which XY is an arm. Then we add the angle which remains over, in each case.

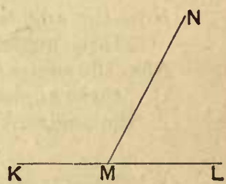
Notice the difference between "are two right angles" and "are equal to two right angles." These figures (5, 5) *are* two 5's, but these figures (4, 6) are *together equal to* two 5's.

Each of the angles AYX, BYX, is called the **Supplement** of the other.

Each of the angles OYX, XYB, is called the **Complement** of the other.

EXERCISES.

1. Quote Definition 10.
2. In the figure of Prop. V. point out any angles which are together equal to two right angles.
3. In this figure, are the angles KMN, NML two right angles?
4. Make an angle equal to half a right angle.
5. What is the size of the Supplement of this angle?
6. What is the size of its Complement?
7. KL is a straight line with MN standing on it. Bisect the angles NML, NMK. Prove that the two bisecting lines are perpendicular to each other.



PROPOSITION XIII. THEOREM. (Second Proof.)

GEN. ENUN.—*The angles which one straight line makes with another straight line on the same side of it, either* are *two right angles, or are* together equal to *two right angles.*

PART. ENUN.—Let the straight line XY make with the straight line AB on one side of it, the angles XYA, XYB.

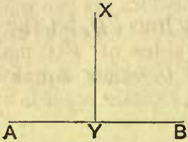


FIG. 1.

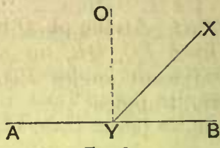


FIG. 2.

We have to prove that these
either are two right angles,
or, are together equal to two right angles.

CASE I.—Where the angle XYA is equal to the angle XYB (Fig. 1).

PROOF—

By Def. 10 these *are* two right angles.

CASE II.—Where the angle XYA is not equal to the angle XYB (Fig. 2).

CONSTRUCTION—

At the point Y draw YO at right angles to AB ....I. 11.

PROOF—

Then, the angles OYA, OYB, *are* two right angles.

Now the angles XYA, XYB are made up of the three angles AYO, OYX, XYB.

Also, the angles OYA, OYB are made up of the same three angles.

$\therefore$ the angles XYA, XYB = the angles OYA, OYB,
i.e., = two right angles.

WHEREFORE, *the angles which one straight line, etc.*

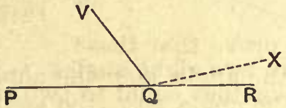
Q.E.D.

Notes and Exercises are at the end of the First Proof.

PROPOSITION XIV. THEOREM.

GEN. ENUN.—*IF at a point in a straight line, two other straight lines on the opposite sides of it make the adjacent angles together equal to two right angles,*
THEN these two straight lines shall be in one and the same straight line.

PART. ENUN.—At the pt. Q in the st. line VQ, let the two st. (Given.) lines PQ, QR, on opposite sides of VQ make the adjacent angles VQP, VQR together equal to two right angles;



Then shall PQ, QR be in the same straight line.

HYPOTHESIS—
Suppose PQR is not a st. line, but that PQX is.

PROOF—
If PQX is a straight line.....................Hyp.
Therefore the angles VQP, VQX = two rt. angles......I. 13.
But the angles VQP, VQR = two rt. angles.....Given.
Therefore VQP, VQR = VQP, VQX.....Ax. 1, 11.
Take away the common angle VQP.
Therefore the angle VQR = the angle VQX.............Ax. 3.
(the whole = its part), which is absurd..Ax. 9.
Therefore the supposition that PQ and QX are in one st. line is false.

In the same way we could prove that no other line but QR can be in the same st. line with PQ.

WHEREFORE PQR is a straight line, and,
If at a point, etc.

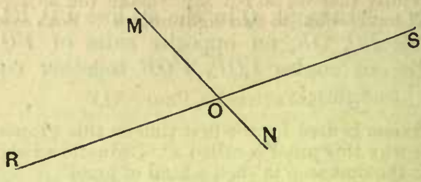
Q.E.D.

For Notes and Exercises see page 51.

PROPOSITION XV. THEOREM.

GEN. ENUN.—*IF two straight lines cut one another,*
THEN the vertically opposite angles are equal to one another.

PART. ENUN.—Let the two straight lines MN and RS cut each other at the point O;



Then shall the angle MOS = the angle RON,
and the angle ROM = the angle NOS.

PROOF—

Because MO falls on the st. line RS,
Therefore the $\angle^s$ MOR, MOS = two rt. $\angle^s$ .... I. 13.

Because RO falls on the st. line MN.
Therefore the $\angle^s$ MOR, RON = two rt. $\angle^s$ .... I. 13.

Therefore the $\angle^s$ MOR, MOS = the $\angle^s$ MOR, RON.... Ax. 1, 11.
Take away the common $\angle MOR$,

Therefore the $\angle MOS$ = the $\angle RON$.................. Ax. 3.

Similarly, we could prove that
the $\angle ROM$ = the $\angle NOS$.

WHEREFORE, *if two straight lines, etc.*

Q.E.D.

COR. 1.—If two straight lines cut one another, the four angles which they make are together equal to four right angles.

COR. 2.—If any number of straight lines meet at a point, all the angles which they make, are together equal to four right angles.

Notes on Prop. XIV.

This Proposition is the converse of Prop. XIII.

In Prop. XIII. we show that $\begin{cases}\text{If } PQR \text{ is a straight line} \\ \text{Then } VQP,\ VQR = \text{two right angles.}\end{cases}$

In Prop. XIV. we show that $\begin{cases}\text{If } \quad VQP,\ VQR = \text{two right angles} \\ \text{Then } PQR \text{ is a straight line.}\end{cases}$

The point Q from which the two lines QP, QR are drawn need not be at the end of the line VQ.

Notice carefully that we do not say "make the adjacent angles two right angles," but "equal to two right angles." Both cases are included in this.

Exercises on Prop. XIV.

1. What Axiom is used for the first time in this Proposition?
2. Explain why this proof is called a "Reductio ad absurdum."
3. What is the first step in such a kind of proof?
4. Suppose the direction of QX came below QR instead of above it. Prove the proposition in this case.
5. Is VQR a right angle?
6. State in your own words the connexion between Props. XIII. and XIV.
7. Why do we say in the Particular Enunciation, "then *shall* PQ, QR, etc."?

Note on Prop. XV.

This Proposition shows that a man who goes along the roads MO, OS, has to make the same "turn" at the corner O, as if he went along RO, ON.

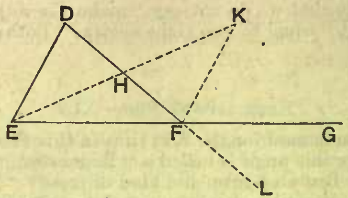
Exercises on Prop. XV.

1. Why is Axiom 11 needed in this Proposition?
2. Prove the Corollaries.
3. Prove that the angle ROM = the angle NOS.
4. If $MO = ON$, and $RO = OS$, and MR, NS be joined, prove that the triangle MOR = the triangle NOS.
5. Bisect the angle MOS by a straight line OX, and produce XO to Y. Prove that OY bisects the angle RON.
6. Can an acute angle be vertically opposite to an obtuse angle? Why?
7. What is the converse of this Proposition?
8. If AOB, XOY are two diameters of a circle, show that AX is equal to BY.

PROPOSITION XVI. THEOREM.

GEN. ENUN.—*IF one side of a triangle be produced,*
THEN the exterior angle is greater than either of the interior opposite angles.

PART. ENUN.—Let *DEF* be a triangle having its side *EF* produced to *G*;



Then shall the exterior angle *DFG* be greater than the interior opposite angle *EDF*, and also greater than the interior opposite angle *DEF*.

CONSTRUCTION—

1. Bisect DF in H, and join EH .... I. 10.
2. Produce EH to K so that $HK = EH$ .... I. 3.
3. Join KF.

PROOF—

1. $\because$ in the $\triangle$s DHE and FHK

$$\text{we have}\left\{\begin{array}{ll} DH = FH & \text{.......... Const.} \\ HE = HK & \text{.......... Const.} \\ \text{and } \angle DHE = \angle FHK & \text{.......... I. 15.} \end{array}\right.$$

$\therefore$ $\angle EDH = \angle KFH$ .... I. 4.

But $\angle DFG$ is greater than the $\angle KFH$.
$\therefore$ $\angle DFG$ is greater than the $\angle EDF$.

2. Similarly, if we produce DF to L, and bisect EF, we could prove that
the $\angle EFL$ is greater than the $\angle DEF$.

But $\angle EFL = \angle DFG$ .... I. 15.
$\therefore$ $\angle DFG$ is greater than $\angle DEF$.

WHEREFORE, *if one side, etc.*

Q.E.D.

NOTES.

In the Construction of this Proposition we bisect that arm of the exterior angle which is not produced, join this point to the opposite angle, and after producing the new line so as to make it as long again, join the new end of it to the vertex of the exterior angle.

The first part of the Proof shows that the exterior angle is greater than that interior opposite angle which is *opposite to the produced side*.

Notice that in the triangles DEH and FKH, D corresponds to F, and E to K.

Of the three interior angles of tho triangle DEF, the two, DEF and EDF, are *opposite* to the exterior angle DFG, and the third, DFE, is interior and adjacent to DFG.

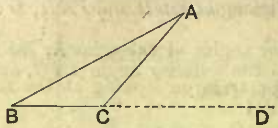
EXERCISES.

1. Which is the greater angle, KFG or KEF, and why?
2. Is the exterior angle of a triangle greater than the *interior adjacent* angle? Draw a figure, and give the reasons for your answer.
3. Why would it not be correct to say "Because DH, HE are equal to KH, HF, each to each, etc."? What is the correct form?
4. Prove, in full, the second part of the Proposition.
5. Prove that $DE = FK$.
6. Prove that the area of the triangle DEF = the area of the triangle KEF.
7. Prove the Proposition when the side DE is produced.
8. Make a list of all the Propositions between I. and XV., which are used, *directly* or *indirectly*, for the Proof of this Theorem. (Begin with the highest number, and work backwards.)

PROPOSITION XVII. THEOREM.

GEN. ENUN.—*Any two angles of a triangle are together less than two right angles.*

PART. ENUN.—Let ABC be a triangle;



Then shall the $\angle^s$ ABC, ACB be less than two rt. $\angle^s$;
and also the $\angle^s$ BAC, ACB be less than two rt. $\angle^s$;
and also the $\angle^s$ BAC, ABC be less than two rt. $\angle^s$.

CONSTRUCTION—

Produce the side BC to D.

PROOF—

$\because$ $\angle ACD$ is the ext^r $\angle$ of the $\triangle$ ABC.

$\therefore$ $\angle ACD$ is gr. than $\angle ABC$.................I. 16.

Add to each the $\angle ACB$.

$\therefore$ $\angle^s$ ACD, ACB are gr. than $\angle^s$ ABC, ACB.....Ax. 4.

But $\angle^s$ $ACD, ACB =$ two rt. $\angle^s$..............I. 13.

$\therefore$ $\angle^s$ ABC, ACB are less than two rt. $\angle^s$.

Similarly we can prove that

$\angle^s$ BAC, ACB are less than two rt. $\angle^s$.

$\angle^s$ BAC, ABC are less than two rt. $\angle^s$.

WHEREFORE, *any two angles, etc.*

Q.E.D.

EXERCISES.

1. Prove that the $\angle^s$ BAC, ACB are less than two right angles.
2. Prove that a triangle cannot have two right angles, or two obtuse angles.
3. Prove that only one perpendicular can be drawn to a straight line from a given external point.
4. How many perpendiculars can be drawn to a straight line from a iven point *in* it?
5. Prove the Proposition without producing a side, by taking a point D in BC, and joining AD. Use Prop. XVI.

AXIOM 12.

If a straight line meet two other straight lines so as to make the two interior angles on the same side of it together less than two right angles, these two straight lines being produced far enough, shall at length meet on that side on which are the angles which are less than two right angles.

Notes.

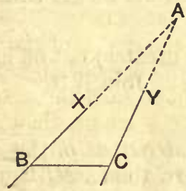
This Axiom will be seen, on looking at the figure below, to be the converse of Prop. XVII.

In the Proposition we prove that

Because ABC is a triangle,
Therefore the $\angle^s$ ABC, ACB are less than two rt. $\angle^s$.

The Axiom states that

If the $\angle^s$ XBC, YCB are less than two rt. $\angle^s$,
Then BX, CY, if produced, will *form a triangle* with BC,
i.e. that they will meet at some point A.



As this so-called Axiom is not quite obvious to our common sense, it has been proposed to substitute for Axiom 12 the following, which is called—

Playfair's Axiom.

"*Two straight lines which intersect cannot both be parallel to*
"*the same straight line.*"

Exercises.

1. Draw a figure to illustrate Playfair's Axiom, and explain it.
2. From Axiom 12 show that if "the two interior angles on the same side of the line" were *greater* than two right angles, the lines, when produced, would meet on the other side.
3. What do you suppose would be the case if the two angles were *equal* to two right angles?
4. Explain the meaning of the word "converse" used above. Give some examples.

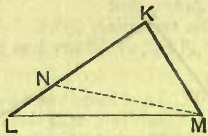
PROPOSITION XVIII. THEOREM.

GEN. ENUN.—*IF one side of a triangle be greater than another side, THEN the angle opposite to the greater side is greater than the angle opposite to the less.*

OR

The greater side of every triangle is opposite to the greater angle.

PART. ENUN.—In the $\triangle$ *KLM* let the side *KL* be greater than *KM*;



Then shall the $\angle KML$ be gr. than the $\angle KLM$.

CONSTRUCTION—

1. From *KL* (the greater) cut off a part $KN = KM$.....I. 3.
2. Join *NM*.

PROOF—

$\because$ *KNM* is the ext^r $\angle$ of the $\triangle$ *LNM*,

$\therefore$ $\angle KNM$ is gr. than $\angle NLM$ (*i.e.* $\angle KLM$)......I. 16.

But $\angle KNM = \angle KMN$ ($\because KN = KM$)........I. 5.

$\therefore$ $\angle KMN$ is gr. than $\angle KLM$.

Much more then is $\angle KML$ gr. than $\angle KLM$.

WHEREFORE, *the greater side, etc.*

Q.E.D.

NOTE.

The second form of the General Enunciation in the XVIIIth and XIXth Propositions is the one given by Euclid; but in this it is difficult to understand what is given and what has to be proved.

EXERCISES.

1. What is the connexion between this Proposition and the Vth?
2. Prove that the great*est* side of every triangle has the great*est* angle opposite to it (*i.e.* greatest of three).
3. In the figure of Prop. V. prove that the angle *KRQ* is greater than the angle *KQR*.

PROPOSITION XIX. Theorem.

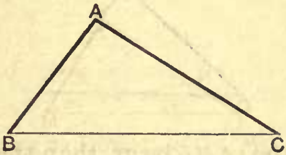
Gen. Enun.—*IF one angle of a triangle be greater than another angle,*

THEN the side opposite to the greater angle is greater than the side opposite to the less.

Or

The greater angle of every triangle is subtended by the greater side.

Part. Enun.—In the $\triangle ABC$ let the $\angle ABC$ be greater than the $\angle ACB$;



Then shall AC be greater than AB.

Proof—

If AC be not greater than AB,

Then AC must be equal to, or less than, AB.

1. If $AC = AB$, then $\angle ACB = \angle ABC$........................I. 5.
   But it does not, $\therefore$ AC is not equal to AB.

2. If AC be less than AB, $\angle ABC$ is less than $\angle ACB$..I. 18.
   But it is not, $\therefore$ AC is not less than AB.

$\therefore$ AC must be greater than AB.

Wherefore, *the greater angle, etc.*

Q.E.D.

Notes.

As in the Vth and VIth Props., we have first the *sides* given equal (I. 5), and then the *angles* given equal (I. 6); so in the XVIIIth and XIXth we have first the *sides* given unequal (I. 18), and then the *angles* given unequal (I. 19). In both cases the latter Proposition is proved by a "Reductio ad Absurdum."

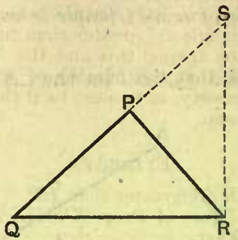
Exercises.

1. What is the connexion between Props. XIX. and VI.?
2. In the figure of Prop. XI. prove that CA is greater than CM.
3. Show that in a right-angled triangle the hypoteneuse is the greatest side.

PROPOSITION XX. THEOREM.

GEN. ENUN.—*Any two sides of a triangle are together greater than the third side.*

PART. ENUN.—Let PQR be the triangle;



Then shall QP, PR be together greater than QR,
PQ, QR be together greater than PR,
QR, RP be together greater than PQ.

CONSTRUCTION—Produce QP to S, so that $PS = PR$,........I. 3.
Join SR.

PROOF—

$\because$ $PS = PR$..........................Const.
$\therefore$ $\angle PSR = \angle PRS$..........................I. 5.
But $\angle QRS$ is greater than $\angle PRS$..................Ax. 9.
$\therefore$ $\angle QRS$ is greater than $\angle PSR$ (*i.e.*, QSR).

$\therefore$ the side QS (*opposite the* $\angle QRS$) is greater than the side QR (*opposite the* $\angle QSR$) ....I. 19.

But $QS = QP$, PR.......................Const.
$\therefore$ the sides QP, PR are together greater than QR.

Similarly, we could prove that
PQ, QR are together greater than PR,
and QR, RP are together greater than QP.

WHEREFORE, *any two sides, etc.*

Q.E.D.

NOTES.

This Proposition shows us that if we have to go from a place Q to a place R, it will be shorter to go along the straight road QR, than by the roads round by P.

Common sense tells us this, and in many editions of Euclid the truth of it is assumed, by giving the following definition of a straight line:—

"A straight line is the shortest distance between two points."

"Proclus, in his commentary, says, that though the truth of it be "manifest to our senses, yet it is science which must give the reason "why two sides of a triangle are greater than the third; but the right "answer to this objection against this and the XXIst, and some other "plain Propositions, is, that the number of Axioms ought not to be "increased without necessity, as it must be if these Propositions be not "demonstrated."—*Simson.*

EXERCISES.

1. Prove that PQ, QR are greater than PR.

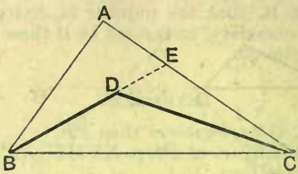
2. Show that, in the figure of Prop. XVIII., KN, NM together are less than KL, LM together.

3. If a point O be taken within a triangle ABC, and OA, OB, OC joined; prove by this Proposition that twice the sum of OA, OB, OC is greater than the sum of the sides of the triangle.

PROPOSITION XXI. THEOREM.

GEN. ENUN.—*IF from the ends of the side of a triangle there be drawn two straight lines to a point within the triangle, THEN these two shall be less than the other two sides of the triangle, but shall contain a greater angle.*

PART. ENUN.—Let ABC be a △, and from the pts. B, C, let the two st. lines BD, CD be drawn to a pt. D within the △;



Then shall 1. BD, DC **be less than** BA, AC.

2. The $\angle BDC$ **be greater than the** $\angle BAC$.

CONSTRUCTION—

Produce BD to meet AC in E.

PROOF—

1. (*a*) In the △ BAE
the sides BA, AE are greater than BE.........I. 20.
Add to each EC,
∴ BA, AE, EC (*i.e.* BA, AC) are gr. than BE, EC.

(*b*) In the △ CED
the sides CE, ED are greater than CD.........I. 20.
Add to each DB,
∴ CE, ED, DB (*i.e.* BE, EC) are gr. than CD, DB.

∴ BA, AC are much greater than CD, DB.

2. ∵ $\angle BDC$ is the ext^r ∠ of the △ CED,
∴ $\angle BDC$ is greater than $\angle DEC$ (or $\angle BEC$).........I. 16.
∵ $\angle BEC$ is the ext^r ∠ of the △ AEB,
∴ $\angle BEC$ is greater than the $\angle BAE$ (or $\angle BAC$).....I. 16.

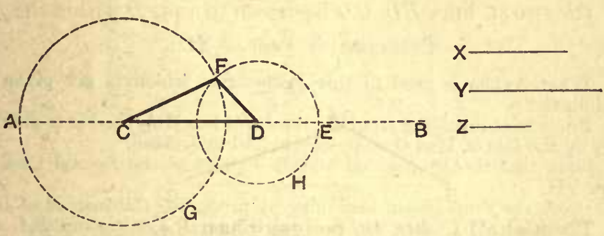
∴ $\angle BDC$ is much greater than $\angle BAC$.

WHEREFORE, *if from the ends, etc.* Q.E.D.

PROPOSITION XXII. Problem.

Gen. Enun.—*To make a triangle of which the sides shall be equal to three given straight lines, any two of which are together greater than the third.*

Part. Enun.—Let X, Y, Z be the three given lines, of which any two are together greater than the third;



It is required to make a △ having its sides respectively equal to X, Y, Z.

Construction—

1. Take a straight line AB terminated at the end A, but unlimited in length towards B.
2. Cut off $AC = X$, $CD = Y$, $DE = Z$ .... I. 3.
3. With centre C and radius CA, describe $\odot^{le}$ AFG.
4. With centre D and radius DE, describe $\odot^{le}$ FEH, and let the $\bigcirc^{ces}$ cut in F.
5. Join CF, FD.

Then shall CFD be the required triangle.

Proof— $\because$ C is cr. of $\odot^{le}$ AFG,

$\therefore$ $CF = CA = X$......... Def. 15. and Const.

$\because$ D is cr. of $\odot^{le}$ FEH,

$\therefore$ $DF = DE = Z$......... Def. 15. and Const.

And we made $CD = Y$ .... Const.

$\therefore$ $CF = X$, $CD = Y$, $DF = Z$.

Wherefore, *a triangle has been made, etc.*

Q.E.F.

Notes on Prop. XXI.

This Proposition shows that if we have to go from a place B to a place C, it is shorter to go by the roads BD, DC, than by BA, AC; and we have to make the sharper turn when we go by A. As in Prop. XX., the fact is quite evident to our common sense.

Notice that we first show that the road by A is longer than the road by E, and then that the road by E is longer than that by D; therefore it is decidedly nearer to go by D than by A.

Exercises on Prop. XXI.

1. What Axiom is used in this Proposition which is not given in Euclid's list?

2. Prove that in sailing from St. Bees Head to Holyhead it is shorter to go by the Isle of Man than by Strangford in Ireland.

3. Show that this Proposition affords a proof of the Second Case of Prop. VII.

4. Prove the Proposition, beginning by producing CD instead of BD.

5. If a point P be taken within a quadrilateral $ABCD$, and PA, PC joined, prove that the sides of the new quadrilateral are together less than the sides of the original one.

Exercises on Prop. XXII.

1. What condition is made about the length of the three given lines in this Proposition, and why?

2. Could triangles be formed with their sides equal to lines of the following lengths respectively?—

(i.)	(ii.)	(iii.)	(iv.)
3 inches.	4 inches.	5 inches.	1 inch.
2 inches.	3 inches.	3 inches.	4 inches.
5 inches.	2 inches.	4 inches.	2 inches.

Draw figures to illustrate your answers, using the same construction as in the Proposition.

3. With the above method of construction how many triangles do we get with their sides the required length?

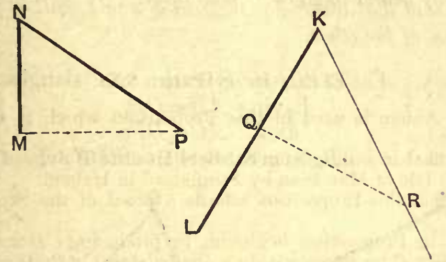
4. What earlier Proposition is a special case of this?

5. How much of the circumferences of the two circles is it necessary to draw?

PROPOSITION XXIII. PROBLEM.

GEN. ENUN.—*At a given point in a given straight line to make a rectilineal angle equal to a given rectilineal angle.*

PART. ENUN.—Let *KL* be the given straight line, *K* the given point, and *MNP* the given angle;



It is required to make at *K* an angle equal to *MNP*.

CONSTRUCTION—

1. In *NM*, *NP* take any pts.* *M*, *P*, and join *MP*.
2. On *KL* make a $\triangle$ *KQR*, having its sides *KQ*, *QR*, *RK* respectively equal to *NM*, *MP*, *PN*......I. 22.

Then shall the $\angle QKR$ = the $\angle MNP$.

PROOF—

$\because$ in the $\triangle^s$ *QKR* and *MNP*

$$\text{we have} \begin{cases} KQ = NM \\ QR = MP \\ RK = PN \end{cases} \text{........................Const.}$$

$\therefore$ the $\angle QKR = \angle MNP$ ....I. 8.

WHEREFORE, *at the given pt. K, etc.* Q.E.F.

EXERCISES.

1. Construct a triangle, having given the length of two of its sides, and the angle between them.

2. Draw the figure of this Proposition, putting in all the construction, as in Prop. XXII.

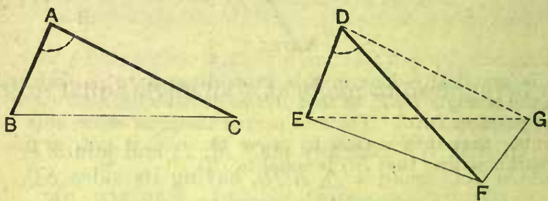
3. Draw any quadrilateral, and construct by Props. XXII., XXIII., another quadrilateral with all its sides and angles equal to those of the first quadrilateral.

* In Practice, *NM*, *MP*, are usually taken of the same length.

PROPOSITION XXIV. THEOREM. (FIRST PROOF.)

GEN. ENUN.—*IF two triangles have*
Two sides of the one equal to two sides of the other, each to each,
But the angles contained by these sides unequal,
THEN
the base of that which has the greater angle shall be greater than the base of the other.

PART. ENUN.—Let ABC, DEF be the two triangles,
having $AB = DE$,
and $AC = DF$,
But $\angle BAC$ greater than $\angle EDF$;



THEN shall BC be greater than EF.

Of the two sides DE, DF, let DE be the one which is not greater than the other.

CONSTRUCTION—

1. At the point D, in the st. line ED, make the $\angle EDG$ equal to the $\angle BAC$ .... I. 23.
2. Make $DG = DF$ or AC .... I. 3.
3. Join GE and GF.

PROOF—

1. In the $\triangle^s$ ABC and DEG,

(To prove that $BC = EG$.)

$\because$ $\begin{cases} BA = ED \text{........................ Given.} \\ AC = DG \text{........................ Const. (2).} \\ \text{and } \angle BAC = \angle EDG \text{........................ Const. (1).} \end{cases}$

$\therefore$ $BC = EG$ .... I. 4.

2. (To prove EG gr. than EF.)

$\because$ $DF = DG$ .... Const. (2).
$\therefore$ $\angle DGF = \angle DFG$ .... I. 5.
But $\angle DGF$ is gr. than $\angle EGF$ .... Ax. 9.
$\therefore$ $\angle DFG$ is gr. than $\angle EGF$,
and $\angle EFG$ is gr. than $\angle DFG$ .... Ax. 9.
$\therefore$ $\angle EFG$ is much gr. than $\angle EGF$.

3. And $\because$ in the $\triangle$ EFG
$\angle EFG$ is gr. than $\angle EGF$ .... Just proved.
$\therefore$ EG is gr. than EF .... I. 19.
But $EG = BC$ .... Proved above.
$\therefore$ BC is gr. than EF.

WHEREFORE, *if two triangles, etc.*

Q.E.D.

NOTES.

What is practically done in this Proposition is to put the triangle ABC on the triangle DEF, so that AB coincides with DE. ABC then takes the position DEG. The first part of the proof shows this.

Remember that it is useless to prove that EG is greater than EF, unless you know first that EG is equal to BC.

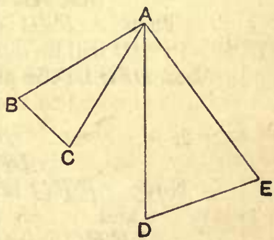
EXERCISES.

1. Draw figures to show the result of the above construction in the case where DE *is* greater than DF.

2. Suppose DE were greater than DF, what must then be done? Draw the figure in this case.

3. To which earlier Proposition is Prop. XXIV. similar? What is the difference between them?

4. In the accompanying figure, ABC and ADE are isosceles triangles, with a common vertex A. Join CD, BE, and prove that BE is greater than CD.



5. If the angle DAE is greater than the angle BAC, prove that CE is greater than BD.

EXERCISES ON PROP. XXV.

1. What is this method of proof called? Why?
2. Of what Proposition is this the converse?

PROPOSITION XXIV. THEOREM. (Second Proof.)

GEN. ENUN.—*IF two triangles have*

Two sides of the one equal to two sides of the other, each to each,

But the angles contained by these sides unequal,

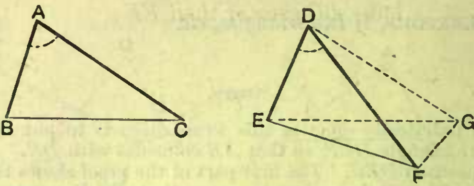
THEN the base of that which has the greater angle shall be greater than the base of the other.

PART. ENUN.—Let ABC, DEF be the two triangles,

having $AB = DE$,

and $AC = DF$,

but $\angle BAC$ greater than $\angle EDF$;



THEN shall BC be greater than EF.

Of the two sides DE, DF, let DE be not gr. than DF.

PROOF—1. $\because$ $AB = DE$ .... Given.

$\therefore$ AB can coincide with DE,

Put the triangle ABC on the triangle DEF, so that AB coincides with DE.

Then $\because$ $\angle BAC$ is greater than $\angle EDF$, ... Given.

$\therefore$ AC will lie outside the triangle DEF.

Let DEG be the position of the triangle ABC.

Join FG.

2. $\because$ $DF = DG$ .... Given.

$\therefore$ $\angle DFG = \angle DGF$ .... I. 5.

Now $\angle EFG$ is greater than $\angle DFG$, and $\angle EGF$ is less than $\angle DGF$ } .... Ax. 9.

$\therefore$ $\angle EFG$ is greater than $\angle EGF$,

$\therefore$ EG is greater than EF .... I. 19.

i.e., BC is greater than EF.

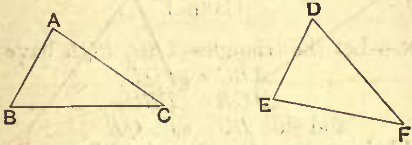
WHEREFORE, *if two triangles, etc.* Q.E.D.

EXERCISES. See page 65.

PROPOSITION XXV. THEOREM.

GEN. ENUN.—*IF two triangles have*
Two sides of the one equal to two sides of the other, each to each,
But the base of the one greater than the base of the other;
THEN the angle contained by the two sides of the triangle with the greater base, shall be greater than the angle contained by the corresponding sides of the other triangle.

PART. ENUN.—Let ABC, DEF be the two triangles,
having $AB = DE$
and $AC = DF$
but BC greater than EF;



THEN shall $\angle BAC$ be greater than $\angle EDF$.

PROOF—
If $\angle BAC$ be not greater than $\angle EDF$, it must be either equal to it, or less than it.

(1) *If* $\angle BAC = \angle EDF$,
Then $BC = EF$ ....I. 4.
But this is not the case,....................Given.
$\therefore$ $\angle BAC$ is not equal to $\angle EDF$.

(2) *If* $\angle BAC$ be less than $\angle EDF$,
Then BC is less than EF....................I. 24.
But neither is this the case....................Given.
Hence $\angle BAC$ must be greater than $\angle EDF$.

WHEREFORE, *if two triangles, etc.* Q.E.D.

NOTE.

To remember which enunciation is the 24th and which the 25th, notice that the 24th corresponds to the 4th, and the 25th to the 8th. Compare the Enunciations of these Propositions one with another.

EXERCISES. See page 65.

PROPOSITION XXVI. THEOREM. (First Proof).

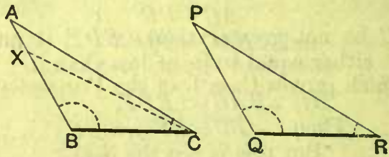
GEN. ENUN.—*IF two triangles have*
Two angles of the one equal to two angles of the other, each to each, and one side equal to one side, viz.,
either (1) *the side adjacent to the equal angles,*
or (2) *a side opposite to an equal angle in each,*

THEN
the remaining sides shall be equal, each to each,
and the third angle of the one triangle shall be equal to the third angle of the other.

CASE I.

PART. ENUN.—Let the triangles ABC, PQR have

$$\angle ABC = \angle PQR,$$
$$\angle ACB = \angle PRQ,$$
and side BC = side QR
(BC being *adjacent* to the $\angle^s$ ABC and ACB).



Then we have to prove that
AB = PQ
AC = PR
and ∠BAC = ∠QPR.

HYPOTHESIS—
Suppose AB is not equal to PQ,
but that AB is greater than PQ.

CONSTRUCTION—
From BA cut off BX equal to PQ ....I. 3
Join CX.

PROOF—

1. In the $\triangle^s$ XBC and PQR,

$$\text{If}\ \left\{\begin{array}{l} XB = PQ \ldots\ldots\ldots\ldots\ldots\ldots\text{Hyp. Const.} \\ \left.\begin{array}{l} BC = QR \\ \text{and } \angle XBC = \angle PQR \end{array}\right\} \ldots\ldots\ldots\ldots\text{Given.} \end{array}\right.$$

$$\therefore\ \angle XCB = \angle PRQ \ldots\ldots\ldots\ldots\ldots\ldots\text{I. 4.}$$

2. But $\angle PRQ = \angle ACB$ .... Given.

$\therefore\ \angle XCB = \angle ACB$ .... Ax. 1.

A part = the whole, which is absurd...... Ax. 9.

$\therefore$ the hypothesis that AB is greater than PQ is incorrect.

Similarly, we could show that AB is not less than PQ,

$\therefore\ AB$ must $= PQ$.

3. In the $\triangle^s$ ABC, PQR,

$$\because\ \left\{\begin{array}{l} AB = PQ \ldots\ldots\ldots\ldots\ldots\ldots\text{Just proved.} \\ BC = QR \ldots\ldots\ldots\ldots\ldots\ldots\text{Given.} \\ \text{and } \angle ABC = \angle PQR \ldots\ldots\ldots\ldots\text{Given.} \end{array}\right.$$

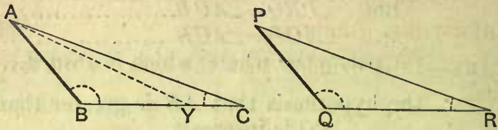
$$\left.\begin{array}{l} \therefore\ AC = PR \\ \text{and } \angle BAC = \angle QPR \end{array}\right\} \ldots\ldots\ldots\ldots\ldots\ldots\text{I. 4.}$$

which proves Case I. of the Proposition.

Case II.

Part. Enun.—Let the triangles ABC, PQR, have

$\angle ABC = \angle PQR$,
$\angle ACB = \angle PRQ$,
and side $AB =$ side PQ.
(AB being *opposite to* one of the equal angles, viz., ACB).



Then we have to prove that
BC=QR,
AC=PR,
∠BAC=∠QPR.

Hypothesis—
Suppose BC is not equal to QR,
but that BC is greater than QR.

Construction—
From BC cut off BY equal to QR.................I. 3.
Join AY.

Proof—
1. In the $\triangle^s$ ABY and PQR,

If $\begin{cases} AB = PQ \text{..........................Given.} \\ BY = QR \text{.................. ..Hyp. Const.} \\ \angle ABY = \angle PQR \text{.......................Given.} \end{cases}$

$\therefore \quad \angle AYB = \angle PRQ$........................I. 4.

2. But $\angle PRQ = \angle ACB$........................Given.
$\therefore \quad \angle AYB = \angle ACB$........................Ax. 1.
i.e. the ext^r of the $\triangle$ $ACY =$ the int^r opp^{te} $\angle$,
Which is impossible.................I. 16.
$\therefore$ the hypothesis that BC is gr. than QR is incorrect.

Similarly we could show that BC is not less than QR.
$\therefore \quad BC$ must $= QR$.

3. In the $\triangle^s\ ABC, PQR,$

$$\because \begin{cases} AB = PQ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\text{Given.} \\ BC = QR \ldots\ldots\ldots\ldots\ldots\ldots\text{Just proved.} \\ \angle ABC = \angle PQR \ldots\ldots\ldots\ldots\ldots\ldots\text{Given.} \end{cases}$$

$$\left.\begin{array}{rl} \therefore & AC = PR \\ \text{and} & \angle BAC = \angle QPR \end{array}\right\} \ldots\ldots\ldots\ldots\ldots\ldots\ldots\text{I. 4.}$$

Which proves Case II. of the Proposition.

WHEREFORE, *if two triangles, etc.* Q.E.D.

COROLLARY—It is evident that the areas of the triangles are equal.

NOTES.

In learning this Proposition, notice that in Case I. BC is given equal to QR, and AB supposed unequal to PQ; in Case II. AB is given equal to PQ, and BC supposed unequal to QR. Also, parts 1 and 3 of the proof are similar in both cases, Prop. IV. being applied first to the hypothetical triangle and the triangle PQR, and then to the two original triangles. In part 2 of the proof, Case I. uses Ax. 9, and Case II. uses I. 16.

Props. IV., VIII., and XXVI. prove the equality of two triangles in all respects, when *three* of their elements are known to be equal, viz.:—

Prop. IV. When two sides and the included angle are given.
Prop. VIII. When three sides are given.
Prop. XXVI. When two angles and one side are given.

In the other cases, viz.:—

When two sides and an angle (not the included one) are given,
or, When three angles are given,

The triangles are not *necessarily* equal, in all respects.
See Appendix XXI.

EXERCISES.

1. Case I. of Prop. XXVI. is the converse of Prop. IV. Show this, putting your answer in a similar form to that given in the Notes on Props. VIII. and XIV.

2. What is the converse of Case II.? Is it true?

3. Write down all the ways in which you can take three of the seven parts of a triangle; and opposite to each set, in another column, write the remaining four parts.

4. In which of the above cases, when the *three* parts are given equal in two triangles, do we know that the remaining four parts will also be equal in both triangles?

5. If from the vertex of an isosceles triangle a straight line be drawn perpendicular to the base, prove that it bisects the base.

6. In Case II., in what other position might the dotted line be drawn? Would this make any difference to the proof?

PROPOSITION XXVI. THEOREM. (Second Proof.)

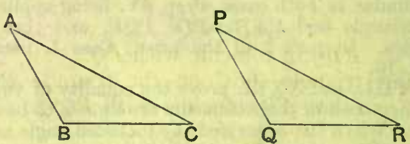
GEN. ENUN.—*IF two triangles have*

Two angles of the one equal to two angles of the other, each to each, and one side equal to one side, viz.,
either (1) *the side adjacent to the equal angles,*
or (2) *a side opposite to an equal angle, in each;*

THEN the remaining sides shall be equal, each to each, and the third angle of the one triangle shall be equal to the third angle of the other.

PART. ENUN.—Let the triangles ABC, PQR have

$\angle ABC = \angle PQR$,
$\angle ACB = \angle PRQ$,
and side BC = side QR.
(BC being *adjacent* to the $\angle^s$ ABC and ACB).



We have to prove that AB = PQ,
AC = PR,
and ∠BAC = ∠QPR.

PROOF—

1. $\because$ $BC = QR$ ....Given.
$\therefore$ BC can coincide with QR.
Put the triangle ABC on the triangle PQR so that the base BC coincides with QR.

2. $\because$ $\angle ABC = \angle PQR$ and $\angle ACB = \angle PRQ$ } ....Given.
$\therefore$ AB will lie on PQ, and AC will lie on PR.
$\therefore$ the point A will fall on the point P.

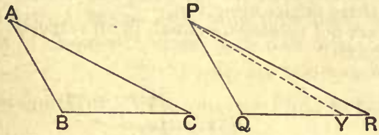
3. Hence the $\triangle ABC$ coincides with the $\triangle PQR$.
$\therefore$ $AB = PQ$, $AC = PR$, and $\angle BAC = \angle QPR$ } ....Ax. 8.
Which proves Case I.

Case II.

Part. Enun.—Let the triangles ABC, PQR, have

$$\angle ABC = \angle PQR,$$
$$\angle ACB = \angle PRQ,$$
$$\text{and side } AB = \text{side } PQ.$$

(AB being *opposite to* one of the equal angles, viz., ACB).



We have to prove that BC=QR,
AC=PR,
∠BAC=∠QPR.

Proof—1. $\because \quad AB = PQ$ .... Given.
$\therefore \quad AB$ can coincide with PQ.
Put the triangle ABC on the triangle PQR, so that AB coincides with PQ.

2. $\because \quad \angle ABC = \angle PQR$ .... Given.
$\therefore \quad BC$ will *lie on* QR.

3. Now, if the pt. C does not lie on the pt. R, it will have some other position along QR, such as Y, and AC will lie in the position PY, and the $\angle ACB$ in the position PYQ.

4. But $\angle PYQ = \angle PRQ$ .... Given.
i.e., the ext^r $\angle$ of the $\triangle PYR$ = the int^r opp^{te} $\angle$.
Which is impossible .... I. 16.
$\therefore$ BC must coincide with QR, and AC with PR.

$$\left.\begin{aligned} \therefore \quad BC &= QR \\ AC &= PR \\ \text{and} \quad \angle BAC &= \angle QPR \end{aligned}\right\} \text{.................... Ax. 8.}$$

Which proves Case II.

Wherefore, *if two triangles, etc.* Q.E.D.

Corollary—The areas of the triangles are also equal.

NOTES.

Props. IV., VIII., and XXVI. prove the equality of two triangles in all respects, when *three* of their elements are known to be equal, viz.:—

Prop. IV. When two sides and the included angle are given.
Prop. VIII. When three sides are given.
Prop. XXVI. When two angles and one side are given.

In the other cases, viz.:—

When two sides and an angle (not the included one) are given,
or, When three angles are given,

The triangles are not *necessarily* equal, in all respects.

See Appendix xxi.

EXERCISES.

1. Case I. of Prop. XXVI. is the converse of Prop. IV. Show this, putting your answer in a similar form to that given in the Notes on Props. VIII. and XIV.
2. What is the converse of Case II.? Is it true?
3. Write down all the ways in which you can take three of the seven parts of a triangle; and opposite to each set, in another column, write the remaining four parts.
4. In which of the above cases, when the *three* parts are given equal in two triangles, do we know that the remaining four parts will also be equal in both triangles?
5. If from the vertex of an isosceles triangle a straight line be drawn perpendicular to the base, prove that it bisects the base.
6. In Case II., in what other position might the dotted line be drawn? Would this make any difference to the proof?

END OF FIRST PART OF BOOK I.

NOTES ON THE SECOND PART OF BOOK I.

The First Part of Book I., which is about straight lines, angles, and triangles, closes with Prop. XXVI. The remaining Propositions in this Book deal with parallels, parallelograms, and areas. Before commencing this part it is necessary to understand the names which are applied to certain angles, which are formed when one straight line cuts two other straight lines. These two lines may either be parallel or not.

Let the straight line $EGHF$ cut (or "fall on") the straight lines AB, CD (parallel in Fig. 1, and not parallel in Fig. 2).

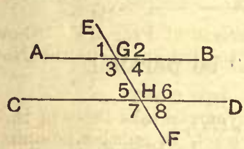


FIG. 1.

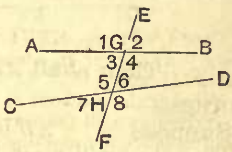


FIG. 2.

We now have 8 angles formed, and for shortness and clearness will number them, and call them by number instead of by letters, *i.e.* the angle EGA is the angle 1, etc.

1. The angles 1, 2, 7, 8 are called *exterior* angles, because they lie *outside* the lines AB, CD.
2. The angles 3, 4, 5, 6 are called *interior* angles, because they lie *between* the lines.
3. The angles 3 and 6 are called *alternate* to each other, *i.e.*, if we consider the angle 3, then 6 is the *alternate* angle.
4. If we take the exterior angle 2, then the angle 6 is said to be *the interior and opposite angle on the same side of the line* (EF).
5. The angles 4 and 6 are called *the two interior angles on the same side of the line.*

EXERCISES.

1. What do we call the angles (i.) AGH, (ii.) FHD, (iii.) CHG, (iv.) AGE, respectively?

2. Which are the two interior angles on the left-hand side of the line?

3. If CHF be an exterior angle, which is "the interior and opposite on the same side of the line"?

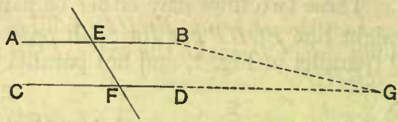
4. Which angle is alternate to CHG?

5. What does Ax. 12 tell you about the above figures?

PROPOSITION XXVII. THEOREM.

GEN. ENUN.—*If a straight line falling on two other straight lines make the alternate angles equal to each other, these two straight lines shall be parallel.*

PART. ENUN.—Let the straight line EF, falling on the two st. lines AB and CD, make the alt^{te} $\angle^s$ AEF and EFD equal;



Then shall AB and CD be parallel.

HYPOTHESIS—

Suppose AB is not $\|^l$ to CD,
Then they will meet, either towards B and D, or towards A and C.

CONSTRUCTION—

Let them be produced towards B, D, and meet in G.

PROOF—

If EBG and FDG are straight lines............Hyp.
$\therefore$ EGF is a $\triangle$, with its side GE produced to A,
$\therefore$ the ext^r $\angle AEF$ is greater than the $\angle EFG$...........I. 16.
But the $\angle AEF$ is also equal to the $\angle EFG$.........Given.
Which is absurd.
$\therefore$ AB and CD do not meet towards B and D.

Similarly we could show that they do not meet towards A and C.

$\therefore$ they are parallel..................Def. 35.

WHEREFORE, *if a straight line, etc.* Q.E.D.

EXERCISES.

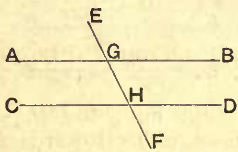
1. Prove the Proposition, when the other pair of alternate angles are given equal.
2. In the figure of Prop. XVI., prove that FK is parallel to DE.
3. What is always the first step in the method of proof used for this Proposition?

PROPOSITION XXVIII. THEOREM.

GEN. ENUN.—*If a straight line falling on two other straight lines make* (1) *the exterior angle equal to the interior and opposite angle on the same side of the line; or* (2) *make the two interior angles on the same side together equal to two right angles; Then these two straight lines shall be parallel.*

PART I.

PART. ENUN.—Let the st. line $EGHF$, falling on the two straight lines AB, CD, make the extr $\angle EGB$ equal to the intr oppte $\angle GHD$;



Then shall AB be parallel to CD.

PROOF— $\because$ $\angle EGB = \angle GHD$ ....Given.
and $\angle EGB = \angle AGH$ ....I. 15.
$\therefore$ $\angle AGH = \angle GHD$ ....Ax. 1.
And these are alternate angles,
$\therefore$ AB is parallel to CD ....I. 27.

PART II.

PART. ENUN.—Let the st. line EF, falling on AB and CD, make the two intr $\angle^s$ BGH, GHD = two right angles;

Then shall AB be parallel to CD.

PROOF— $\because$ the $\angle^s$ BGH and GHD = two rt. $\angle^s$.........Given.
and the $\angle^s$ BGH and AGH = two rt. $\angle^s$ ....I. 13.
$\therefore$ the $\angle^s$ BGH and GHD = the $\angle^s$ BGH and AGH...Ax. 1, 11.
Take away the common angle BGH,
$\therefore$ $\angle AGH = \angle GHD$ ....Ax. 3.
And these are alternate angles,
$\therefore$ AB is parallel to CD ....I. 27.

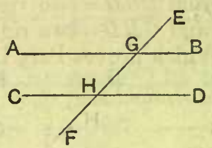
WHEREFORE, *if a straight line, etc.* Q.E.D.

EXERCISES. See page 79.

PROPOSITION XXIX. THEOREM. (First Proof.)

GEN. ENUN.—*If a straight line fall on two parallel straight lines it makes* (1) *the alternate angles equal to one another;*
(2) *the exterior angle equal to the interior and opposite angle on the same side;*
(3) *and also the two interior angles on the same side together equal to two right angles.*

PART. ENUN.—Let AB and CD be parallel, with $EGHF$ falling on them;



Then shall

(1) $\angle AGH =$ alt^{te} $\angle GHD$.
(2) the ext^r $\angle EGB =$ int^r opp^{te} $\angle GHD$.
(3) the int^r $\angle^s$ BGH and $GHD =$ two right angles.

PART I.

HYPOTHESIS—

Suppose $\angle AGH$ does not equal $\angle GHD$,
but that $\angle AGH$ is greater than $\angle GHD$.

PROOF— Add to each the $\angle BGH$,
Then the $\angle^s$ AGH, BGH are gr. than $\angle^s$ BGH, GHD..Ax. 4.
But $\angle^s$ AGH and $BGH =$ two rt. $\angle^s$............I. 13.
$\therefore$ $\angle^s$ BGH and GHD are less than two rt. $\angle^s$.
$\therefore$ AB and CD will meet towards B and D.....Ax. 12.
But they are ||^l, and cannot meet............Given.

$\therefore$ the supposition that $\angle AGH$ is not equal to the $\angle GHD$ is erroneous.
$\therefore$ the $\angle AGH = \angle GHD$.
Which proves Part I.

PART II.

$\because \quad \angle AGH = \angle GHD$ ....Just proved.
and $\angle AGH = \angle EGB$ ....I. 15.
$\therefore \quad \angle EGB = \angle GHD$ ....Ax. 1.
Which proves Part II.

PART III.

$\because \quad \angle EGB = \angle GHD$ ....Just proved.
Add to each the $\angle BGH$,
$\therefore$ the $\angle^s$ EGB and BGH = the $\angle^s$ BGH and GHD...Ax. 2.
But the $\angle^s$ EGB and BGH = two right angles.........I. 13.
$\therefore$ the $\angle^s$ BGH and GHD = two right angles.........Ax. 1.
Which proves Part III.

WHEREFORE, *if a straight line, etc.*

Q.E.D.

NOTE.

This is the Proposition for which Euclid invented Axiom 12. The proof of it is regarded as "the great difficulty of Elementary Geometry."

Axiom 12 is not really a "self-evident truth," or "Common Notion," though we have seen that it much more nearly becomes so when regarded as the converse of Prop. XVII.

The method of proof by Playfair's Axiom is given on the next page.

EXERCISES ON PROP. XXVIII.

1. Prove the Proposition, with different angles to those given here, *e.g.* the angles CHF and AGH for Part I.

2. How many different pairs of angles might be given for Part I.? How many for Part II.?

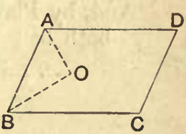
3. When are two straight lines parallel?

EXERCISES ON PROP. XXIX.

1. Of what Propositions is this the converse? Show the truth of your answer by stating in each case what is given, and what is to be proved.

2. Prove that the diagonal of a parallelogram makes equal angles with opposite sides.

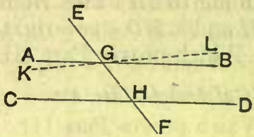
3. $ABCD$ is a parallelogram, and AO, BO the bisectors of the angles DAB, ABC, respectively. Prove, by this Proposition, that the angles OAB, OBA are together equal to a right angle.



PROPOSITION XXIX. THEOREM. (Second Proof.)

GEN. ENUN.—*If a straight line fall on two parallel straight lines it* (1) *makes the alternate angles equal to one another;*
(2) *the exterior angle equal to the interior and opposite angle on the same side;*
(3) *and also, the two interior angles on the same side together equal to two right angles.*

PART. ENUN.—Let AB and CD be parallel, with $EGHF$ falling on them;



Then shall

(1) $\angle AGH$ = alt^te^ $\angle GHD$,

(2) **the ext^r^** $\angle EGB$ = **int^r^ opp^te^** $\angle GHD$,

(3) **the int^r^** $\angle$^s^ BGH **and** GHD = **two rt. angles.**

HYPOTHESIS—

Suppose $\angle AGH$ does not equal $\angle GHD$.

CONSTRUCTION—

At the pt. G in GH make the $\angle KGH = \angle GHD$.........I. 23.
Produce KG to L.

PART I.

PROOF— *If* $\angle KGH$ = alt^te^ $\angle GHD$.............Hyp. Const.
$\therefore$ KL is $\|^l$ to CD.......................I. 27.
But AB is $\|^l$ to CD.....................Given.
And, by Playfair's axiom, AB and KL cannot both be $\|^l$ to CD ....I. 17. Note.
$\therefore$ $\angle AGH$ is not unequal to $\angle GHD$,
i.e., $\angle AGH$ is equal to $\angle GHD$.
Which proves Part I.

PART II.

$\because \angle AGH = \angle GHD$ .... Just proved.

and $\angle AGH = \angle EGB$ .... I. 15.

$\therefore \angle EGB = \angle GHD$ .... Ax. 1.

Which proves Part II.

PART III.

$\because \angle EGB = \angle GHD$ .... Just proved.

Adding to each the $\angle BGH$,

$\therefore$ the $\angle^s$ EGB and BGH = the $\angle^s$ BGH and GHD... Ax. 2.

But the $\angle^s$ EGB and BGH = two right angles...... I. 13.

$\therefore$ the $\angle^s$ BGH and GHD = two right angles.... Ax. 1.

Which proves Part III.

WHEREFORE, *if a straight line, etc.*

Q.E.D.

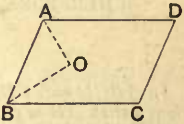
NOTE.

For Playfair's Axiom, see page 55.

EXERCISES.

1. Of what Propositions is this the converse? Show the truth of your answer by stating in each case what is given, and what is to be proved.

2. Prove that the diagonal of a parallelogram makes equal angles with opposite sides.



3. $ABCD$ is a parallelogram, and AO, BO the bisectors of the angles DAB, ABC, respectively. Prove, by this Proposition, that the angles OAB, OBA are together equal to a right angle.

4. Prove that the angle CHG is equal to the angle HGB.

5. Prove that all the angles of a parallelogram are together equal to four right angles.

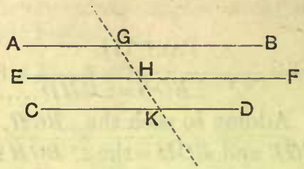
6. Prove that the angle AGE is equal to the angle FHD.

7. Prove that the angles EGB and FHD are together equal to two right angles.

PROPOSITION XXX. THEOREM.

GEN. ENUN.—*Straight lines which are parallel to the same straight line are parallel to each other.*

PART. ENUN.—Let AB, CD be each parallel to EF;



Then shall AB be parallel to CD.

CONSTRUCTION—

Draw a straight line GHK, cutting AB, EF, CD, in G, H, K, respectively.

PROOF—

1. $\because$ AB is $\|^l$ to EF .... Given.
   $\therefore$ $\angle AGH = \text{alt}^{te} \angle GHF$ .... I. 29.

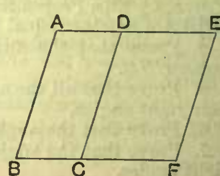
2. $\because$ EF is $\|^l$ to CD .... Given.
   $\therefore$ $\text{ext}^r \angle GHF = \text{int}^r \text{ opp}^{te} \angle HKD$ .... I. 29.

3. And $\angle GHF$ also $= \angle AGH$ .... Proved above.
   $\therefore$ $\angle AGH = \angle GKD$ .... Ax. 1.
   and these are alternate angles,
   $\therefore$ AB is $\|^l$ to CD .... I. 27.

WHEREFORE, *straight lines, etc.*

Q.E.D.

EXERCISES.

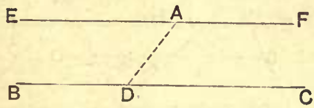


1. $ABCD$ and $CDEF$ are two parallelograms. Prove that AB is parallel to EF.
2. Why is this not sufficient to prove that $ABFE$ is a parallelogram?
3. Show that when EF lies between AB and CD, as in the figure above, that it can be proved that AB and CD cannot meet, by Definition 35 only.

PROPOSITION XXXI. PROBLEM.

GEN. ENUN.—*To draw a straight line through a given point parallel to a given straight line.*

PART. ENUN.—Let A be the given point, and BC the given straight line;



It is required to draw through A a st. line $\|^l$ to BC.

CONSTRUCTION—

1. In BC take a point D, and join AD.
2. At the pt. A in the st. line AD, make an $\angle EAD$ equal to the $\angle ADC$ ....I. 23.
3. Produce EA to F.

Then shall EF be parallel to BC.

PROOF—

$\because$ $\angle EAD =$ alt^{te} $\angle ADC$ ....Const

$\therefore$ EF is $\|^l$ to BC......................I. 27.

WHEREFORE, *a straight line EAF has been drawn through A, parallel to BC.*

Q.E.F.

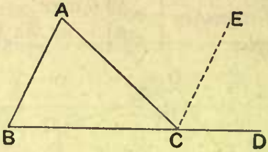
EXERCISES.

1. In Part 2 of the Construction what other angle might be taken instead of ADC?
2. Do this Problem without using Prop. XXIII.
3. How does the above construction indicate the correctness of Playfair's Axiom?
4. Draw a triangle, and then make another triangle outside it, whose sides are parallel to those of the first.
5. Show that all the angles of the first triangle are equal to those of the second.

PROPOSITION XXXII. THEOREM.

GEN. ENUN.—*If a side of any triangle be produced, the exterior angle is equal to the two interior and opposite angles; and the three interior angles of every triangle are equal to two right angles.*

PART. ENUN.—Let ABC be a $\triangle$ with BC produced to D;



Then shall (1) the $\angle ACD$ = the $\angle^s ABC$ and BAC.

(2) the $\angle^s$ ABC, BCA, CAB = two right $\angle^s$.

CONSTRUCTION—

Through C draw CE $\|^l$ to AB ....I. 31.

PART I.

PROOF—

1. $\because$ AC meets the $\|^{ls}$ AB, CE,
 $\therefore$ $\angle ACE$ = alt^{te} $\angle BAC$ ....I. 29.

2. $\because$ BD meets the $\|^{ls}$ AB, CE,
 $\therefore$ ext^r $\angle ECD$ = int^r opp^{te} $\angle ABC$ ....I. 29.

3. $\therefore$ Whole $\angle ACD = \angle^s$ ABC and BAC.........Ax. 2.
 Which proves Part I.

PART II.

1. $\because$ $\angle ACD = \angle^s$ ABC and BAC......Just proved.
 Add to each the $\angle BCA$,
 $\therefore$ $\angle^s$ ACD and $BCA = \angle^s$ ABC, BAC, BCA.......Ax. 2.

2. But the $\angle^s$ ACD, BCA = two rt. $\angle^s$...........I. 13.
 $\therefore$ also the $\angle^s$ ABC, BAC, BCA = two rt. $\angle^s$......Ax. 1.
 Which proves Part II.

WHEREFORE, *if a side of a triangle, etc.* Q.E.D.

NOTE.

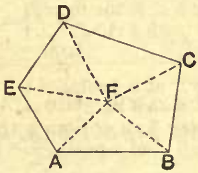
This Proposition is an extension of the 16th.

EXERCISES.

1. Prove that in Exercise 3, Prop. XXIX., the angle AOB is a right angle.
2. What is the size of the angle of an equilateral triangle?
3. In a right-angled isosceles triangle, what is the size of each of the base angles?

COROLLARY I.*

All the interior angles of any rectilineal figure, together with four right angles $=$ *Twice as many right angles as the figure has sides.*



Let $ABCDE$ be the figure.

CONSTRUCTION—

Take any pt. F within the figure, and join it to all the angular points.

We now evidently have *as many triangles as the figure has sides.*

PROOF—

1. $\because$ the $\angle^s$ of one $\triangle = 2$ rt. $\angle^s$ ....I. 32.

$\therefore$ the $\angle^s$ of all the $\triangle^s = \left\{\text{Twice as many rt. } \angle^s \text{ as there are } \triangle^s\right.$,

i.e. $= \left\{\text{Twice as many rt. } \angle^s \text{ as the figure has sides.}\right.$

2. Also, the $\angle^s$ of all the $\triangle^s$ *make up* $\left\{\text{The } \angle^s \text{ of the figure, } \textit{with} \text{ the } \angle^s \text{ at } F.\right.$

i.e. $= \left\{\text{The } \angle^s \text{ of the figure, } \textit{with } 4 \text{ rt. } \angle^s\right.$....I. 15. Cor. 2.

3. But, the $\angle^s$ of all the $\triangle^s = \left\{\text{Twice as many rt. } \angle^s \text{ as the figure has sides}\right.$...Proved in 1.

$\therefore$ the $\angle^s$ of the figure *with* 4 rt. $\angle^s = \left\{\text{Twice as many rt. } \angle^s \text{ as the figure has sides}\right.$..Ax. 1.

Q.E.D.

* For Alternative Proof see page 103.

PARTICULAR CASE.

Let the figure have 5 sides,
Then on joining F to the angular points, we have 5 triangles.

1. $\because$ The 3 $\angle^s$ of one $\triangle = 2$ rt. $\angle^s$....................I. 32.

$\therefore$ the 15 $\angle^s$ of all the $\triangle^s = 10$ rt. $\angle^s$.

2. Now, $\left.\begin{array}{c}\text{the 15 } \angle^s \text{ of all} \\ \text{the } \triangle^s\end{array}\right\}$ *make up* $\left\{\begin{array}{l}\text{The 5 } \angle^s \text{ of the figure,} \\ \textit{with} \text{ the } \angle^s \text{ at } F.\end{array}\right.$

i.e. = $\left\{\begin{array}{l}\text{The 5 } \angle^s \text{ of the figure,} \\ \textit{with } \text{4 rt. } \angle^s.\end{array}\right.$

3. But also, the 15 $\angle^s$ of all the $\triangle^s = 10$ rt. $\angle^s$ ....Proved above.

$\therefore$ $\left.\begin{array}{c}\text{the 5 } \angle^s \text{ of the figure} \\ \textit{with} \text{ 4 rt. } \angle^s\end{array}\right\} = 10$ rt. $\angle^s$.

Hence, $\left.\begin{array}{c}\text{the 5 } \angle^s \text{ of a pentagon} \\ \textit{with} \text{ 4 rt. } \angle^s\end{array}\right\} = 10$ rt. $\angle^s$.

Take away 4 rt. $\angle^s$ from both sides,
$\therefore$ the 5 $\angle^s$ of a pentagon $= 6$ rt. $\angle^s$.

So, the 6 $\angle^s$ of a hexagon *with* 4 rt. $\angle^s = 12$ rt. $\angle^s$.
$\therefore$ the 6 $\angle^s$ of a hexagon $= 8$ rt. $\angle^s$.

And the 8 $\angle^s$ of an octagon *with* 4 rt. $\angle^s = 16$ rt. $\angle^s$.
$\therefore$ the 8 $\angle^s$ of an octagon $= 12$ rt. $\angle^s$.

And so on.

EXERCISES.

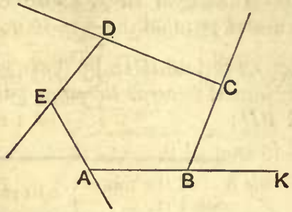
1. Prove that the angles of any quadrilateral figure are together equa to four right angles.
2. What is the size of an angle of a heptagon, all of whose angles are equal?
3. Show the truth of this Corollary for a triangle.
4. Can you have "a five-sided figure which has all its sides equal, and all its angles right angles"? Why?
5. If all the interior angles of a rectilineal figure are equal to sixteen right angles, how many sides has it?

EXERCISES ON COR. II.

1. Draw the figure, producing BA in the direction B to A.
2. Demonstrate the truth of Cor. II. of Prop. XV., by a practical method similar to that mentioned in the Notes.
3. Show that a similar practical proof may be given of the 1st Corollary of Prop. XXXII. Begin by placing the straight edge along AB, and turn it so that the straight edge lies along each side in turn, and comes alternately *inside* and *outside* of the figure.
4. Show *practically* that both Corollaries are true for a figure of twelve sides. What is such a figure called?

Corollary II.

All the exterior angles of any rectilinear figure are together equal to four right angles.



Let $ABCDE$ be the figure, and CBK one of the exterior angles.

NOTE—

To each interior angle there is an exterior angle adjacent, and there are as many interior angles as sides.

PROOF—

Now the int^r $\angle ABC$, with its adjac^t ext^r $\angle CBK$ = two right $\angle^s$ ....I. 13.

Hence, $\left.\begin{array}{l}\text{all the int}^r \angle^s \\ \textit{with} \text{ the ext}^r \angle^s\end{array}\right\} = \left\{\begin{array}{l}\text{Twice as many rt. } \angle^s \\ \text{as the figure has sides.}\end{array}\right.$

$= \left\{\begin{array}{l}\text{all the interior } \angle^s \\ \quad \textit{with } 4 \text{ rt. } \angle^s\end{array}\right.$ ....Cor. I.

Take away the int^r $\angle^s$ from both sides.

$\therefore$ the exterior $\angle^s = 4$ rt. $\angle^s$ ....Ax. 3.

Q.E.D.

Notes.

The sides of the figure must be produced in the same direction of rotation; *i.e.* if AB be produced in the direction A to B, we must produce BC in the direction B to C, CD in the direction C to D, and so on, going round the figure always in the same direction.

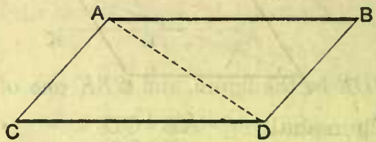
A *practical* demonstration of this Corollary may be shown, by laying a "straight edge" along one of the sides of the figure, and outside the figure, and then turning it round so as to coincide with each side in turn. When it comes back to the original side it will be found to have turned completely round, *i.e.* to have passed through *four right angles.*

From this we see that if a man walks completely round a piece of ground with any number of sides, he will have turned right round once, and have faced, in turn, all the Four Cardinal Points of the Compass. At each corner he deviates from his former path, by an angle, or "turn," which is an exterior angle of the figure.

PROPOSITION XXXIII. THEOREM.

GEN. ENUN.—*The straight lines which join the extremities of two equal and parallel straight lines, towards the same parts, are themselves equal and parallel.*

PART. ENUN.—Let AB and CD be two equal and parallel straight lines, joined *towards the same parts* by the straight lines AC and BD;



Then shall (1) AC **be equal to** BD,
(2) AC **be parallel to** BD.

CONSTRUCTION— Join AD.

PROOF— $\because$ AB is $\|^l$ to CD ....Given.
$\therefore$ $\angle BAD = \text{alt}^{te}\ \angle ADC$ ....I. 29.

In the $\triangle^s$ BAD and CDA

$$\because \begin{cases} BA = CD \text{ ....Given.} \\ AD = DA \text{ ....Common.} \\ \text{and } \angle BAD = \angle CDA \text{ ....Proved above.} \end{cases}$$

$$\left.\begin{array}{r} \therefore\ AC = BD \\ \text{and } \angle ADB = \angle DAC \end{array}\right\} \text{ ....I. 4.}$$

And these are alternate $\angle^s$,
$\therefore$ AC is $\|^l$ to BD ....I. 27.

And we proved $AC = BD$.

WHEREFORE, *the straight lines, etc.* Q.E.D.

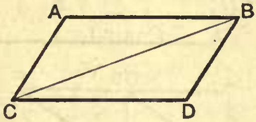
EXERCISES.

1. What is meant by "towards the same parts"? Draw two equal and parallel straight lines, and join their ends *not towards the same parts.*
2. Of what kind does this Proposition prove the figure $ABCD$ to be?
3. What is the converse of this Proposition?

PROPOSITION XXXIV. THEOREM.

GEN. ENUN.—*The opposite sides and angles of a parallelogram are equal to one another, and the diameter bisects the parallelogram, that is, divides it into two equal parts.*

PART. ENUN.—Let $ACDB$ be a parallelogram, and BC a diameter;



Then shall $\mathbf{AB = CD}$

$\mathbf{AC = BD}$

$\mathbf{\angle BAC = \angle BDC}$

$\mathbf{\angle ABD = \angle ACD}$

and area ABC = area BCD.

PROOF—

1. $\because$ BC meets the $\|^{ls}$ AB and CD.........Def. 36.

$\therefore$ $\angle ABC =$ alt^{te} $\angle BCD$..................I. 29.

$\because$ BC meets the $\|^{ls}$ AC and BD.........Def. 36.

$\therefore$ $\angle DBC =$ alt^{te} $\angle BCA$..................I. 29.

$\therefore$ the whole $\angle ABD =$ the whole $\angle ACD$.......Ax. 2.

Which proves one pair of opposite $\angle^s$ equal.

2. In the $\triangle$s ABC and BCD,

$$\because \begin{cases} \left.\begin{aligned} \angle ABC &= \angle BCD \\ \angle ACB &= \angle CBD \end{aligned}\right\} \text{............Just proved.} \\ \text{and } BC \text{ is common.} \end{cases}$$

$$\therefore \left.\begin{aligned} AB &= CD \\ AC &= BD \\ \angle BAC &= \angle BDC \end{aligned}\right\} \text{.....................I. 26.}$$

and area $ABC =$ area BCD...............I. 26. Cor.

WHEREFORE, *the opposite sides, etc.*

EXERCISES.

1. Prove this Proposition by joining AD.
2. Prove that the triangle ABC is equal to the triangle ACD in area.

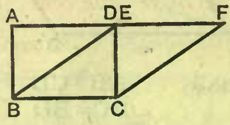
PROPOSITION XXXV. THEOREM.

GEN. ENUN.—*Parallelograms on the same base, and between the same parallels, are equiareal to each other.*

PART. ENUN.—Let the $\square^{\text{grams}}$ $ABCD$, $EBCF$ be on the same base BC, and between the same $\|^{\text{ls}}$ AF, and BC;

Then shall $\square^{\text{gram}}$ $ABCD$ be equiareal to $\square^{\text{gram}}$ $EBCF$.

CASE I.



Where the points D and E coincide.

PROOF—

$\because$ $\square^{\text{gram}}$ $ABCD$ is double of the $\triangle DBC$ }
and $\square^{\text{gram}}$ $EBCF$ is also double of the $\triangle DBC$ } ....I. 34.
$\therefore$ the $\square^{\text{gram}}$ $ABCD = \square^{\text{gram}}$ $EBCF$............Ax. 6.

CASE II.

Where the points D and E do not coincide.

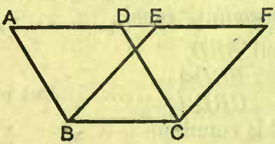


FIG 1.

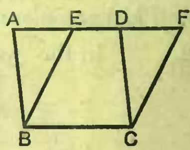


FIG. 2.

PROOF—

1. $\because$ $ABCD$ is a parallelogram,
$\therefore$ $AD = BC$ ....I. 34.
Similarly $EF = BC$,
$\therefore$ $AD = EF$ ....Ax. 1.

Now add DE (Fig. 1), *or* take away DE (Fig. 2).

$\therefore$ { the whole $AE =$ the whole DF (Fig. 1).........Ax. 2.
{ *or*, the rem$^{\text{dr}}$ $AE =$ the rem$^{\text{dr}}$ DF (Fig. 2)..........Ax. 3.

2. Now, in the $\triangle^s$ EAB, FDC

$$\because \begin{cases} EA = FD \text{ ....Just proved.} \\ AB = DC \text{ ....I. 34.} \\ \text{and } \angle EAB = \angle FDC \text{ ....I. 29.} \end{cases}$$

$\therefore \quad \triangle EAB = \triangle FDC$ ....I. 4.

3. Take the $\triangle FDC$ from the trapezium $ABCF$, and we have left the $\square^{gram}$ $ABCD$.
Take the $\triangle EAB$ from the same trapezium, and we have left the $\square^{gram}$ $EBCF$.
$\therefore$ the rem^{dr} $ABCD$ = the rem^{dr} $EBCF$..........Ax. 3.

WHEREFORE, *parallelograms on the same base, etc.*

Q.E.D.

NOTES.

The Enunciation of this Proposition is usually given thus:—
"*Parallelograms on the same base and between the same parallels, are equal to each other.*"

As this use of the word "equal" is sometimes confusing to beginners, who often take it to imply "equal in all respects"; the word "equiareal" has been introduced to point out that the parallelograms are to be regarded as equal in *area* only. The same symbol is used as for "is equal to."

A similar remark applies to the succeeding Propositions.

If from a point in one of the parallels AF, BC a perpendicular be drawn to the other, the length of this line is called the **altitude** of the parallelograms.

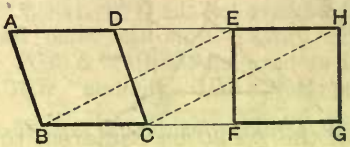
EXERCISES.

1. In what special way is Axiom 3 used at the end of this Proposition?
2. Draw the figure of Case I., and cut it up into pieces, so as to show by laying the pieces on one another that the parallelograms are equiareal.
3. In the same figure prove that if $ABCD$ is a square, $EBCF$ cannot be a rhombus. What is it? (Use I. 19.)
4. What relation does the figure of Case I. bear to the figures in Case II?
5. Prove the Proposition without using I. 4.
6. Prove that parallelograms of equal altitude, on the same base, and on the same side of it, are equiareal.

PROPOSITION XXXVI. THEOREM.

GEN. ENUN.—*Parallelograms on equal bases, and between the same parallels, are equiareal to one another.*

PART. ENUN.—Let $ABCD$, and $EFGH$ be $\square^{\text{grams}}$ on equal bases BC, and FG, and between the same $\|^{\text{s}}$ AH and BG;



Then shall $\square^{\text{gram}}$ $ABCD$ be equiareal to $\square^{\text{gram}}$ $EFGH$.

CONSTRUCTION—

Join BE, and CH.

PROOF—

1. $\because$ $BC = FG$........................Given.
And $FG = EH$..........................I. 34.
$\therefore$ $BC = EH$..........................Ax. 1.
And $\because$ BC and EH are equal and parallel,
$\therefore$ BE, and CH which join them, are parallel......I. 33.
$\therefore$ $EBCH$ is a $\square^{\text{gram}}$....................Def. 36.

2. $\because$ $EBCH$, and $ABCD$ are on the same base BC, and between the same $\|^{\text{ls}}$ AH and BC...Given.
$\therefore$ $\square^{\text{gram}}$ $EBCH = \square^{\text{gram}}$ $ABCD$ ....I. 35.

And $\because$ $EBCH$, and $EFGH$ are on the same base EH, and between the same $\|^{\text{ls}}$ EH, BG.....Given.
$\therefore$ $\square^{\text{gram}}$ $EBCH = \square^{\text{gram}}$ $EFGH$ ....I. 35.

Hence, $\square^{\text{gram}}$ $ABCD = \square^{\text{gram}}$ $EFGH$...............Ax. 1.

WHEREFORE, *parallelograms on equal bases, etc.*

Q.E.D.

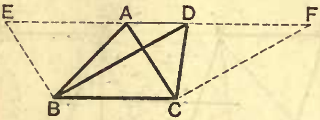
EXERCISES.

1. What is the first fact that it is necessary to prove in this Proposition, and why?
2. What is the converse of this Proposition?

PROPOSITION XXXVII. THEOREM.

GEN. ENUN.—*Triangles on the same base and between the same parallels, are equiareal to one another.*

PART. ENUN.—Let ABC, and DBC be $\triangle^s$ on the same base BC, and between the same $\|^{ls}$ AD and BC;



Then shall the $\triangle ABC$ be equiareal to the $\triangle DBC$.

CONSTRUCTION—

1. Produce AD both ways to E and F.
2. Through B draw BE $\|^l$ to AC } ....I. 31.
3. Through C draw CF $\|^l$ to BD }

Then $EBCA$ and $DBCF$ are $\square^{grams}$..........Def. 36.

PROOF—

$\because$ the $\square^{grams}$ EC* and BF are on the same base BC, and between the same $\|^{ls}$ BC and EF........Given.

$\therefore$ $\square^{gram} EC = \square^{gram} BF$ ....I. 35.

But $\triangle ABC$ is half the $\square^{gram} EC$ } ....I. 34.
And $\triangle DBC$ is half the $\square^{gram} BF$ }

$\therefore$ $\triangle ABC = \triangle DBC$ ....Ax. 7.

WHEREFORE, *triangles on the same base, etc.*

Q.E.D.

NOTE.

If a straight line be drawn from the vertex of a triangle perpendicular to the base, or the base produced, the length of it is called the **Altitude** of the triangle.

EXERCISES.

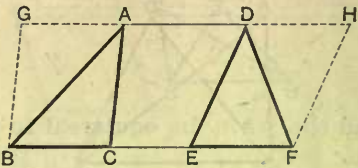
1. Prove the Proposition by drawing a straight line through B parallel to CD, and through C parallel to BA.
2. Can the two triangles be equal in all respects? Does I. 7 forbid this?
3. Prove that triangles of equal altitude, on the same base, are equiareal.

* A parallelogram is sometimes referred to by the letters standing at two opposite angles. Thus $AEBC$ is called the parallelogram EC, or AB.

PROPOSITION XXXVIII. THEOREM.

GEN. ENUN.—*Triangles on equal bases and between the same parallels are equiareal to each other.*

PART. ENUN.—Let ABC and DEF be $\triangle^s$ on equal bases BC and EF, and between the same $\|^{ls}$ AD and BF;



Then shall $\triangle ABC$ be equiareal to $\triangle DEF$.

CONSTRUCTION—

1. Produce AD both ways to G and H.
2. Through B draw BG $\|^l$ to CA } ....I. 31.
3. Through F draw FH $\|^l$ to ED }

Then $GBCA$ and $DEFH$ are $\square^{grams}$..........Def. 36.

PROOF—

$\because$ the $\square^{grams}$ GC and DF are on equal bases, BC, EF, and between the same $\|^{ls}$ GH and BF...Given

$\therefore$ $\square^{gram}$ $GC = \square^{gram}$ DF..................I. 36.

But $\triangle ABC$ is half $\square^{gram}$ GC } ....I. 34.
And $\triangle DEF$ is half $\square^{gram}$ DF }

$\therefore$ $\triangle ABC = \triangle DEF$..................Ax. 7.

WHEREFORE, *triangles on equal bases, etc.*

Q.E.D.

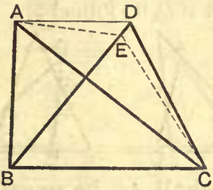
EXERCISES.

1. In what other way might the parallel lines be drawn in the Construction?
2. Prove that the triangle DFH = the triangle ABC.
3. AB is a straight line, bisected at C. Take any point D, outside the line, and join it to A, B, C. Prove that the triangle DAC is equal to the triangle DBC.
4. In this Proposition, must the bases of the triangles be in the same straight line

PROPOSITION XXXIX. THEOREM.

GEN. ENUN.—*Equiareal triangles on the same base, and on the same side of it, are between the same parallels.*

PART. ENUN.—Let ABC and DBC be equiareal $\triangle^s$ on the same base BC, and let AD be joined;



Then shall AD be parallel to BC.

HYPOTHESIS—

Suppose AD is not $\|^l$ to BC.

CONSTRUCTION—

1. Through A draw AE $\|^l$ to BC, meeting BD in E...I. 31.
2. Join CE.

PROOF— *If* $\triangle^s$ ABC, EBC, are on the same base BC, and between the same $\|^{ls}$ AE, BC..........Hyp.

$\therefore$ $\triangle ABC = \triangle EBC$..........................I. 37.

But $\triangle ABC = \triangle DBC$..........................Given.

$\therefore$ $\triangle EBC = \triangle DBC$..........................Ax. 1.

A part = the whole, which is absurd..Ax. 9.

$\therefore$ AE is not $\|^l$ to BC.

And in the same way we could show that no other line through A, but AD, is $\|^l$ to BC,

$\therefore$ AD is $\|^l$ to BC.

WHEREFORE, *equiareal triangles, etc.* Q.E.D.

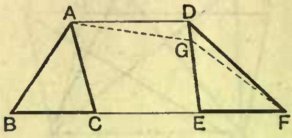
EXERCISES.

1. Which Proposition is the converse of this?
2. Prove the Proposition, supposing AE lies above instead of below AD.
3. Why are the words in the Enunciation "on the same side of it," necessary?

PROPOSITION XL. THEOREM.

GEN. ENUN.—*Equiareal triangles, on equal bases in the same straight line, and on the same side of it, are between the same parallels.*

PART. ENUN.—Let ABC, DEF be equiareal $\triangle^s$ on equal bases BC, EF, in the same straight line BF, and on the same side of it, and let AD be joined;



Then shall AD be parallel to BF.

HYPOTHESIS—Suppose AD is not $\|^l$ to BF.

CONSTRUCTION—

1. Through A draw AG $\|^l$ to BF, meeting DE in G...I. 31.
2. Join FG.

PROOF—*If* ABC and GEF are on equal bases BC, EF,
and between the same $\|^{ls}$ AG, BF..........Hyp.
$\therefore$ $\triangle ABC = \triangle GEF$......................I. 38.
But $\triangle ABC = \triangle DEF$......................Given.
$\therefore$ $\triangle GEF = \triangle DEF$......................Ax. 1.
A part = the whole, which is absurd..........Ax. 9.
$\therefore$ AG is not parallel to BF.

In the same way we can show that no other line through A but AD is $\|^l$ to BF.
$\therefore$ AD is $\|^l$ to BF.

WHEREFORE, *equiareal triangles, etc.*

Q.E.D.

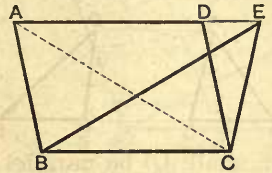
EXERCISES.

1. Prove the Proposition when AG is drawn above AD instead of below it.
2. Are equiareal triangles on equal bases in the same straight line always between the same parallels?
3. Use this Proposition to prove that
"Equiareal parallelograms on equal bases in the same straight "line, and on the same side of it, are between the same parallels."

PROPOSITION XLI. THEOREM.

GEN. ENUN.—*If a parallelogram and a triangle be on the same base, and between the same parallels, the parallelogram shall be double of the triangle.*

PART. ENUN.—Let the ▯^{gram} $ABCD$, and the $\triangle EBC$ be on the same base BC, and between the same $\|^{ls}$ BC, AE;



Then shall *ABCD* be double of *EBC*.

CONSTRUCTION— Join AC.

PROOF—

$\because$ the $\triangle^s$ ABC, EBC, are on the same base BC, and between the same $\|^{ls}$ AE, BC............Given.

$\therefore$ $\triangle ABC = \triangle EBC$......................I. 37.

But ▯^{gram} $ABCD$ is double of $\triangle ABC$............. I. 34.

$\therefore$ ▯^{gram} $ABCD$ is double of $\triangle EBC$.

WHEREFORE, *if a parallelogram, etc.*

Q.E.D.

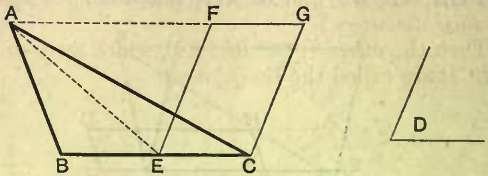
EXERCISES.

1. Draw a rhombus, and make a right-angled triangle half its size.
2. Having done this, make a triangle equal to the rhombus.
3. Show that one triangle which can be thus made equal to the rhombus will be isosceles.
4. If a parallelogram and a triangle are on *equal* bases and between the same parallels, the parallelogram will be double of the triangle.
5. If a parallelogram $ABCD$ and a triangle EBC are on the same base BC, and the parallelogram is double of the triangle, prove that E will lie in AD, or AD produced.
6. What relation does this last exercise bear to Prop. XLI.?
7. Prove the Proposition without joining AC, by drawing through B a parallel to CE.

PROPOSITION XLII. PROBLEM.

GEN. ENUN.—*To describe a parallelogram that shall be equal to a given triangle, and have one of its angles equal to a given rectilineal angle.*

PART. ENUN.—Let ABC be the given $\triangle$, and D the given $\angle$;



It is required to make a ▯gram equal in area to the $\triangle ABC$, and having an $\angle$ equal to D.

CONSTRUCTION—

1. Bisect BC in E, and join AE................I. 10.
2. At the pt. E in the line CE make $\angle CEF = \angle D$......I. 23.
3. Through A draw AG ||ˡ to BC, and through C draw CG ||ˡ to EF...................................I. 31.

Then $FECG$ is a ▯gram..................Def. 36.

It shall be the ▯gram required.

PROOF—

$\because$ $BE = EC$.......................Const.

$\therefore$ $\triangle ABE = \triangle AEC$..................I. 38.

$\therefore$ $\triangle ABC$ is double of $\triangle AEC$.

But also ▯gramFC is double of $\triangle AEC$.. ......I. 41.

$\therefore$ ▯gram$FC = \triangle ABC$..................Ax. 6.

and it has an $\angle FEC = D$.................Const.

WHEREFORE, *a parallelogram has been made, etc.*

Q.E.F.

EXERCISES.

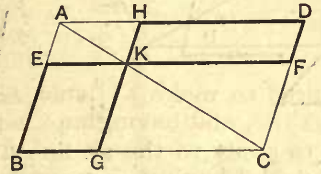
1. Make an oblong equal to a given equilateral triangle.
2. Draw a triangle, and make another double the size of the first.
3. Make a right-angled triangle equal in area to a given rhomboid.

PROPOSITION XLIII. THEOREM.

GEN. ENUN.—*The complements of the parallelograms which are about the diameter of any parallelogram, are equiareal to one another.*

PART. ENUN.—Let $ABCD$ be a $\square^{\text{gram}}$, with diameter AC; and EH, GF, $\square^{\text{grams}}$ *about* AC; (*i.e., which have parts of* AC *for their diameters.*)

Then the other $\square^{\text{grams}}$ BK, KD, which *complete* the figure $ABCD$ are called the *Complements.*



The comp^{t} *BK* shall be equiareal to the comp^{t} *KD*.

PROOF—

$\because$ BD is $\square^{\text{gram}}$,
$\therefore$ $\triangle ABC = \triangle ADC$ ....I. 34.

$\because$ EH is $\square^{\text{gram}}$,
$\therefore$ $\triangle AEK = \triangle AHK$ ....I. 34.

$\because$ GF is $\square^{\text{gram}}$,
$\therefore$ $\triangle KGC = \triangle KFC$ ....I. 34.

$\therefore$ $\triangle^{s}$ AEK, KGC tog^{r} $=$ $\triangle^{s}$ AHK, KFC tog^{r}...Ax. 2.

Take away the $\triangle^{s}$ AEK, KGC from the $\triangle ABC$ and we have left the comp^{t} BK.

Take away the $\triangle^{s}$ AHK, KFC from the $\triangle ADC$ and we have left the comp^{t} KD.

$\therefore$ comp^{t} $BK =$ comp^{t} KD ....Ax. 3.

WHEREFORE, *the complements, etc.*

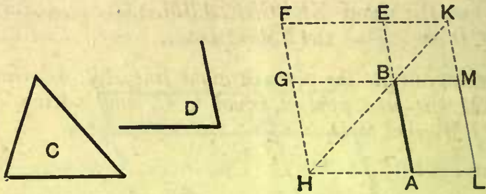
Q.E.D.

For Exercises see page 101.

PROPOSITION XLIV. PROBLEM.

GEN. ENUN.—*To a given straight line to apply a parallelogram, which shall be equal to a given triangle, and have one of its angles equal to a given rectilineal angle.*

PART. ENUN.—Let AB be the given st. line, C the given triangle, and D the given angle;



It is required to apply to the st. line AB a ▱gram equal to △C, and having an ∠ equal to D.

CONSTRUCTION—

1. Make a ▱gram $BEFG$ equal to the △C, and having an ∠EBG = ∠D .... I. 42.
   And place it so that BE may be in the same st. line with AB.
2. Produce FG to H.
3. Through A draw AH ||l to BG or EF .... I. 31.
4. Join HB.

SUBSIDIARY PROOF.*
Now, ∵ AH is ||l to EF .... Const.
∴ the ∠s AHF, HFE = two rt. ∠s .... I. 29.
∴ the ∠s BHF, HFE are less than two rt. ∠s.
∴ HB, FE will meet if produced towards B and E .... Ax. 12.

5. Produce HB, FE to meet in K.
6. Through K draw KL ||l to EA or FH .... I. 31.
7. Produce GB, HA to meet KL in M and L.

Then shall $ALMB$ be the ▱gram required.

* See Note.

PROOF—

1. $\because$ *HLKF* is a $\square^{\text{gram}}$ with *HK* as diam^{r}, and *LB*, *BF* are the complements of *AG*, *ME* (*the* $\square^{\text{grams}}$ *about the diameter*),

$\therefore$ comp^{t} $LB =$ comp^{t} BF .... I. 43.

But $BF = \triangle C$ .... Const.

$\therefore$ $LB = \triangle C$ .... Ax. 1.

2. And $\because$ $\angle EBG = \angle D$ .... Const.

and $\angle EBG = \angle ABM$ .... I. 15.

$\therefore$ $\angle ABM = \angle D$ .... Ax. 1.

WHEREFORE, *to the given straight line AB, a parallelogram AM has been applied, equal to C, and having an angle ABM equal to D.*

Q.E.F.

NOTE.

If Playfair's Axiom be used instead of Ax. 12, the Subsidiary Proof will run as follows :—

$\because$ *GB* and *HB* intersect.
and *GB* is $\|^{l}$ to *FE* .... Const.
$\therefore$ *HB* is not $\|^{l}$ to *FE*, and if produced will meet it...Playfair's Axiom.

EXERCISES.

1. Construct the parallelogram as above, *accurately*, with ruler and compasses.
2. Which other angle of the parallelogram *LB* is equal to *D*?
3. Prove that a square is a parallelogram, and explain why the *converse* of this is not true.
4. Does this Problem enable you to make an oblong equal to a given triangle? Give your reasons, and draw a figure.

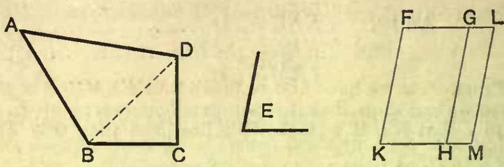
EXERCISES ON PROP. XLIII.

1. Show that *BK*, *KD*, are parallelograms.
2. Prove that the parallelogram $EC =$ the parallelogram *HC*.
3. Draw two parallelograms which have adjoining parts of the same line for diameters, and then draw the corresponding complements.
4. Prove that all the parallelograms in the figure of Prop. XLIII. are equiangular to each other.

PROPOSITION XLV. PROBLEM.

GEN. ENUN.—*To describe a parallelogram equal to a given rectilineal figure, and having an angle equal to a given rectilineal angle.*

PART. ENUN.—Let $ABCD$ be the given rect[l] figure, and E the given angle;



It is required to describe a parallelogram equal to $ABCD$, and having an angle equal to E.

CONSTRUCTION—

1. Join BD.
2. Describe ▯[gram] $FKHG$, equal to $\triangle ABD$, and having an $\angle FKH$ equal to $\angle E$ .... I. 42.
3. Apply to the st. line GH the ▯[gram] GM, equal to $\triangle BCD$, and having $\angle GHM$ equal to $\angle E$ .... I. 44.

Then shall $FKML$ be the parallelogram required.

PROOF—

1. $\because \angle FKH = \angle E$ .... Const.
and $\angle GHM = \angle E$ .... Const.
$\therefore \angle FKH = \angle GHM$ .... Ax. 1.
Add to each the $\angle KHG$.
$\therefore \angle^s FKH, KHG = \angle^s KHG, GHM$ .... Ax. 2.
But $\angle^s FKH, KHG$ = two right angles......... I. 29.
$\therefore \angle^s KHG, GHM$ = two right angles....... Ax. 1.
$\therefore$ *KHM is a straight line* .... I. 14.

2. $\because FG$ is $\|^l$ to KM .... Const.
$\therefore \angle FGH$ = alt[te] $\angle GHM$ .... I. 29.
Add to each the $\angle HGL$.
$\therefore \angle^s FGH, HGL = \angle^s GHM, HGL$ .... Ax. 2.
But $\angle^s GHM, HGL$ = two right angles......... I. 29.
$\therefore \angle^s FGH, HGL$ = two right angles....... Ax. 1.
$\therefore$ *FGL is a straight line*............................ I. 14.

3. $\because$ *FK* and *LM* are each $\|^{l}$ to *GH*............Const.
$\therefore$ *FK* is $\|^{l}$ to *LM*.........................I. 30.
$\therefore$ *FKML is a parallelogram*.........................Def. 36.

4. $\because$ $\square^{\text{gram}}$ $FH = \triangle ABD$.................. Const.
and $\square^{\text{gram}}$ $GM = \triangle BCD$..................Const.
$\therefore$ $\square^{\text{gram}}$ $FM =$ fig. $ABCD$..................Ax. 2.
and $\angle FKM = \angle E$.........................Const.

WHEREFORE, *a parallelogram FM has been made, etc.*

Q.E.F.

NOTES.

In this Proposition we have first to prove that *FKML* is a parallelogram, before we can show that it is the parallelogram required. Hence we first prove that *KHM* is a straight line, and then that *FGL* is a straight line.

In the case given the rectilineal figure has four sides. If it had more than four sides we should have to divide it into three or more triangles, and then construct parallelograms equal to those triangles successively.

This method of dividing a rectilineal figure into triangles suggests another proof of the First Corollary of Prop. XXXII.

CONSTRUCTION—Join one angular point (A) to all the others, except the two on either side of it.

PROOF—$\because$ Each side except the two which meet in A, belongs to a separate triangle.

$\therefore$ This gives a number of triangles which is *two less* than the number of sides.

Now, All the $\angle^{s}$ of the $\triangle^{s}$ *make up* The $\angle^{s}$ of the figure.

And The $\angle^{s}$ of 2 other $\triangle^{s}$ = 4 rt. $\angle^{s}$...............I. 32.

$\therefore$ $\left.\begin{array}{l}\text{The } \angle^{s} \text{ of as many } \triangle^{s} \\ \text{as the figure has sides}\end{array}\right\} = \left\{\begin{array}{l}\text{The } \angle^{s} \text{ of the figure} \\ \textit{with } 4 \text{ rt. } \angle^{s}\end{array}\right.$.......Ax. 3.

$\therefore$ $\left.\begin{array}{l}\text{The } \angle^{s} \text{ of the figure} \\ \textit{with } 4 \text{ rt. } \angle^{s}\end{array}\right\} = \left\{\begin{array}{l}\text{Twice as many rt. } \angle^{s} \\ \text{as the figure has sides}\end{array}\right.$...I. 32.

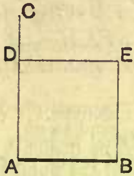
EXERCISES.

1. Construct the figure of this Problem accurately with ruler and compasses, showing all the working.
2. How do you know that *FG* is parallel to *KH*?
3. In Part 2 of the Proof we say "Because *FG* is parallel to *KM*." Why not "Because *FL* is parallel to *KM*"?
4. Show how to make a parallelogram equal to a given six-sided figure, and prove your construction.
5. Suppose we were required to make a parallelogram equal to a given 12-sided figure, what is the smallest number of triangles into which it could be divided?

PROPOSITION XLVI. PROBLEM.

GEN. ENUN.—*To describe a square upon a given straight line.*

PART. ENUN.—Let AB be the given straight line;



It is required to describe a square on AB.

CONSTRUCTION—

1. From A draw $AC \perp^r$ to AB..............I. 11.
2. From AC cut off AD equal to AB..............I. 3.
3. Through D draw $DE \parallel^l$ to AB, and through B draw $BE \parallel^l$ to AD, meeting DE in E..........I. 31.

Then shall $ABED$ be a square.

PROOF—

1. $\because$ $ABED$ is a $\square^{\text{gram}}$..................Const.

$$\left.\begin{aligned} \therefore\ AB &= DE \\ \text{and } AD &= BE \end{aligned}\right\}\text{......................I. 34.}$$

But $AB = AD$........................Const.

$\therefore$ the $\square^{\text{gram}}$ has all its sides equal.

2. $\because$ AD meets the $\parallel^{ls}$ DE, AB..............Const.

$\therefore$ $\angle^s$ BAD, ADE = two rt. $\angle^s$..............I. 29.

But $\angle BAD$ is a rt. $\angle$...................Const.

$\therefore$ $\angle ADE$ is a rt. $\angle$,

and the opp^{te} $\angle^s$ of $ABED$ are equal............I. 34.

$\therefore$ each of the $\angle^s$ ABE, BED is a rt. $\angle$.

$\therefore$ $ABED$ has all its sides equal and all its angles right angles,

$\therefore$ it is a square..........................Def. 30.

WHEREFORE, *a square has been described, etc.*

Q.E.F.

COROLLARY—

From this proof it is evident that every parallelogram which has one right angle has all its angles right angles.

NOTES.

The expression used in Arithmetic and Algebra, "the square *of* a quantity," is closely connected with the Geometrical Expression, "the square *on* AB."

"The square *of* x" means "x multiplied by x," and "the square *of* 6" means "6 multiplied by 6," and so on.

Now, if the straight line AB be 6 inches long, then the area of "the square *on* AB" is 36 inches, *i.e.*, the number of square inches in it is 6^2 (or the square *of* 6).

Similarly, if AB is x inches long, the number of square inches in "the square *on* AB" is x^2 (the square *of* x).

EXERCISES.

1. Show that not more than one square can be described on the same straight line, and on the same side of it.

2. Hence, or otherwise, show that if two straight lines are equal the squares described on them are also equal.

3. Prove the Corollary.

4. How does the Corollary of this Proposition enable us to modify Euclid's definitions of a square and an oblong?

5. Draw a straight line 3 inches long, describe a square on it, and draw straight lines to divide it so as to show the number of square inches it contains.

6. How many square feet are there in the square on a line

(i.) 2 feet long,
(ii.) 5 feet long,
(iii.) 17 feet long?

Draw figures to illustrate the truth of your answers.

7. Prove that if two squares are equal, a side of the one is equal to a side of the other.

8. If a square contains 49 square inches, what is the length of the line on which it is described?

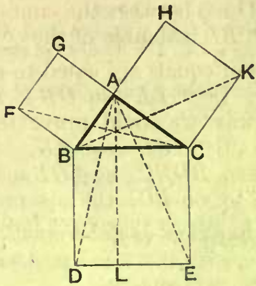
9. Explain in your own words what is the connexion between "the square of a quantity" and "the square on a line." Which of the two expressions do you suppose is derived from the other?

PROPOSITION XLVII. THEOREM.

GEN. ENUN.—*In any right-angled triangle,*

The square which is described on the side subtending the right angle	*is equal to*	*the squares described on the sides which contain the right angle.*

PART. ENUN.—Let ABC be a right-angled $\triangle$, with BAC the rt. angle;



Then shall sq. on BC = sqs. on BA, AC.

CONSTRUCTION—

1. On AB, BC, CA describe the sqs. BE, BG, AK respectively....................I. 46.
2. Through A draw $AL \parallel^l$ to BD, or CE, meeting DE in L....................I. 31.
3. Join FC, BK, AD, AE.

PROOF—

1. $\because$ $\angle BAC$ is a rt. $\angle$....................Given.
   and $\angle BAG$ is a rt. $\angle$....................Const.
   $\therefore$ the adj. $\angle^s$ BAC and BAG = two rt. $\angle^s$.
   $\therefore$ GAC is a st. line....................I. 14.
   Similarly BAH is a st. line.

2 Now $\because$ $\angle DBC = \angle FBA$....................Ax. 11.
   Adding to each the $\angle ABC$,
   $\therefore$ $\angle ABD = \angle FBC$....................Ax. 2.

And in the $\triangle^s$ ABD, FBC,

$\because \begin{cases} AB = FB \text{..........Const.} \\ BD = BC \text{..........Const.} \\ \angle ABD = \angle FBC \text{..........Just proved.} \end{cases}$

$\therefore$ $\triangle ABD = \triangle FBC$..........I. 4.

3. $\because$ $\triangle FBC$ and sq. GB are on the same base FB and between same $\|^{ls}$ FB, GC,

$\therefore$ Sq. GB is double of $\triangle$ FBC..........I. 41.

And $\because$ $\triangle ABD$ and $\square^{gram}$ BL are on the same base BD and between the same $\|^{ls}$ BD, AL,

$\therefore$ $\square^{gram}$ BL is double of the $\triangle ABD$..........I. 41.

But doubles of equals are equal to each other,

$\therefore$ $\square^{gram}$ BL = sq. GB..........Ax. 6.

In the same way we could prove that

$\square^{gram}$ CL = sq. AK.

$\therefore$ the whole sq. $BDEC$ = sqs. GB and AK..........Ax. 2.

i.e. the sq. on BC = the sqs. on BA, AC.

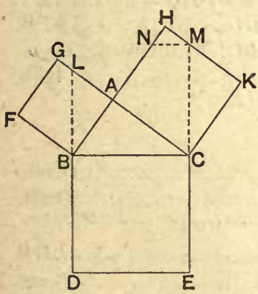
WHEREFORE, *in any right-angled triangle, etc.*

Q.E.D.

NOTE.

This Proposition—one of the most important in Book I.—is said to have been discovered by Pythagoras, who flourished about 500 B.C., 200 years before Euclid.

EXERCISES.



1. Draw the figure given here carefully. (DB and EC are produced to L and M respectively, and MN is parallel to BC.)
Cut the smaller squares out with a sharp penknife, and divide them into parts by cutting along the dotted lines. Now show that the five pieces of the two smaller squares can be made to fit exactly on the space of the larger one.

2. Show that Prop. xlvii. holds for a right-angled triangle whose sides are 3, 4, and 5 inches respectively.

3. Prove that the parallelogram CL = the square AK.

4. Prove that BAH is a straight line.

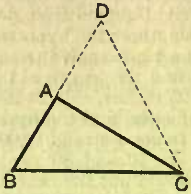
5. Given two straight lines, find a third straight line, the square on which shall be equal to the sum of the squares on the other two.

6. Make a square which shall be double of a given square.

PROPOSITION XLVIII. THEOREM.

GEN. ENUN.—*If the square described on one of the sides of a triangle be equal to the squares described upon the other sides of it, the angle contained by these two sides is a right angle.*

PART. ENUN.—Let ABC be a $\triangle$ such that sq. on BC = sqs. on BA and AC;



Then shall the angle BAC be a right angle.

CONSTRUCTION—

1. From A draw $AD \perp^r$ to AC .... I. 11.
2. Make $AD = AB$, and join DC................ I. 3.

PROOF—

1. $\because$ $DA = AB$.......................... Const.

$\therefore$ sq. on DA = sq. on AB.

Add to each the sq. on AC.

$\therefore$ sqs. on DA, AC = sqs. on BA, AC .... Ax. 2.

But sqs. on DA, AC = sq. on DC.................... I. 47.

And sqs. on BA, AC = sq. on BC.................. Given.

$\therefore$ sq. on DC = sq. on BC................... Ax. 1.

$\therefore$ $DC = BC$.

2. In the $\triangle^s$ ABC, DBC,

$\because \begin{cases} BA = DA \text{.........................Const.} \\ AC = AC \text{.......................Common.} \\ BC = CD \text{..................Just proved.} \end{cases}$

$\therefore$ $\angle BAC = \angle DAC$......................... I. 8.

But $\angle DAC$ is a rt. $\angle$....................... Const.

$\therefore$ $\angle BAC$ is a rt. $\angle$.

WHEREFORE, *if the square described, etc.*

Q.E.D.

NOTE.

Avoid the mistake of saying, in the Construction (1), "produce BA to D." AD must be drawn perpendicular to AC.

EXERCISES.

1. Prove that the triangle whose sides are 12, 16, 20 inches respectively is right-angled.
2. Which of the following triangles are right-angled?—(1) Sides 4, 5, 6 inches; (2) Sides 5, 12, 13 inches; (3) Sides 7, 8, 10 inches.
3. If the two sides of a right-angled triangle are 6 and 8 inches long respectively, what is the length of the hypoteneuse?
4. If the hypoteneuse of a right-angled triangle is 13 inches long, and one of the sides 12 inches, find the length of the other side.
5. Given hypoteneuse = 25 inches, side 24 inches, find the other side.
6. Given sides 9 and 12 inches, find the hypoteneuse.
7. Would it be possible to get a straight 100 yards course on a rectangular field, whose sides are 70 and 80 yards respectively? Draw a figure to show where you would put the course.
8. If a man stands 120 yards away from the foot of a tower whose height is 150 feet, what distance in a straight line will the *top* of the tower be from the place where he is standing. Draw a sketch to represent it.

END OF BOOK I.

BOOK II.

DEFINITIONS.

1. **A Rectangle**, or right-angled parallelogram, is said to be contained by any two of the straight lines which contain one of the right angles.

2. In every parallelogram the figure composed of *either* of the parallelograms about the diameter, together with the two complements, is called a **Gnomon.**

NOTES.

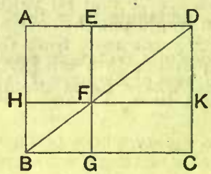


FIG. 1.

1. In the above figure the rectangle $ABCD$ is said to be contained by AB and AD, or by AD and DC, etc., and is shortly referred to as "the rectangle $AB.AD$." This method of denoting the rectangle is in conformity with Algebraic usage; for, if the length of AD be a inches, and the length of AB, b inches, then the area of the figure AC will be $a.b$ square inches, or, more shortly, ab square inches. So, if AD represents 8 feet, and AB 5 feet, then the area of the rectangle $ABCD$ represents $(5 \times 8 =)$ 40 square feet.

2. The parallelogram HG, with the two complements AF and FC, form the gnomon EHC, or AGK, denoted by the letters at opposite angles of the parallelograms which compose it. Similarly, the parallelogram EK, with the complements AF and FC, form the gnomon AKG or CEH.

EXERCISES.

1. What is the name given to a rectangle in Book I.?
2. In the above figure, which are the rectangles contained by

(i.) AE and AH.	(iv.) BG and EF.
(ii.) BG and HB.	(v.) DK and CG.
(iii.) KF and DK.	(vi.) CG and BH.

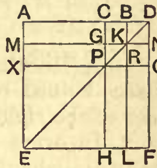
3. In the figure of I. 47, construct the rectangle contained by AB and AC, and that in I. 48 contained by AD and AC.

4. If the side of a square be 4 feet, what is its area?

5. What are the areas of the rectangles whose adjacent sides are respectively

(i.) 3 inches and 4 inches.
(ii.) 6 feet and 3 feet.
(iii.) 8 inches and 1 yard.
(iv.) 1 pole and 60 feet.

Show the truth of your answer to (i.) by a figure.



6. Is a square a rectangle? Is a rectangle a square?

7. How many gnomons are there in Fig. 2? Name them.

8. In this figure point out the rectangles contained by

(i.) *DN* and *MG*.
(ii.) *PX* and *RL*.
(iii.) *EX* and *OR*.
(iv.) *BD* and *GP*.

9. If the area of a rectangle be 24 square inches, and one of its sides be 3 inches, what is the length of the other side? Show by a figure.

10. What is the length of the side of a square whose area is 144 square feet?

11. If two sides of a rectangle be a feet, and b feet respectively, what is its area?

12. Are any of the sides of a rectangle equal?

13. If x^2 represent the area of a square *ABCD*, what does x represent?

14. Given two straight lines *AB* and *CD*, construct the rectangle which they represent.

15. Find the area of a square whose perimeter is 22 yards.

16. What is the area of a rectangle whose perimeter is 22 yards?

17. What else must be given in the above question?

18. Find each of the other sides of a rectangle—

(i.) When the area is 1 acre, and one side is 88 yards;
(ii.) When the area is 15 square inches, and two adjacent sides together are 8 inches.

19. If $a+b$ be the length of a straight line *AB*, what is the area of the square on *AB*?

20. If c be the length of another line *KL*, what area is represented (i.) by c^2; (ii.) by $(a+b)c$?

21. If *ABCD* be a rectangle draw the rectangle which is contained by half *AB* and half *BC*.

22. If two opposite sides of a rectangle be each 4 inches, what do you know about its area?

23. If *ABCD* be a rectangle, by what sides may it be said to be contained?

24. If two opposite sides of a square be each 4 inches, what do you know about its area?

25. Show, by trial, that a square whose perimeter is 24 inches has a larger area than an oblong with the same perimeter. In which cases do you find the area of the oblong the greatest.

26. In Figure 1, by what are the rectangles *EFKD* and *AEFH* said to be contained? How many answers can be given for each rectangle?

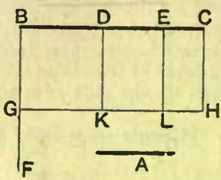
27. Why are *AF* and *FC* called "complements"? (Fig. 1.)

PROPOSITION I. THEOREM.

GEN. ENUN.—*If there be two straight lines, one of which is divided into any number of parts,*

Then the rectangle contained by the two straight lines is equal to *the rectangles contained by the undivided line, and the several parts of the divided line.*

PART. ENUN.—Let A and BC be two straight lines, one of which, BC, is divided into parts at the points D, E;



Then shall
the rect. $A \cdot BC =$ **the rects.** $A \cdot BD$, $A \cdot DE$, **and** $A \cdot EC$.

CONSTRUCTION—

1. From B, draw $BF \perp^r$ to BC .... I. 11.
2. From BF, cut off BG equal to A .... I. 3.
3. Through G, draw $GH \parallel^l$ to BC .... I. 31.
4. Through D, E, C, draw DK, EL, CH, respectively, $\parallel^l$ to BG, and meeting GH, in K, L, H .... I. 31.

PROOF—

Rect. $A \cdot BC =$ the figure BH $(\because BG = A)$ .... Const.
$=$ the figs. BK, DL, EH.
$=$ the rects. $BG \cdot BD$; $DK \cdot DE$; $EL \cdot EC$.
$=$ the rects. $A \cdot BD$; $A \cdot DE$; $A \cdot EC$.
$(\because BG = DK = EL = A)$ .... I. 34.

WHEREFORE, *If there be two straight lines, etc.*

Q.E.D.

ALGEBRAIC PROOF.

Let $A = x$, $BD = a$, $DE = b$, $EC = c$.
Then $BC = a + b + c$.

We have to show that

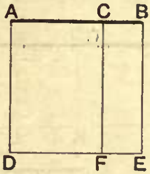
$$x \cdot (a + b + c) = xa + xb + xc.$$

Which is evident. Q.E.D.

PROPOSITION II. Theorem.

Gen. Enun.—*If a straight line be divided into any two parts, Then the rectangles contained by the whole and each of the parts are equal to the square on the whole line.*

Part. Enun.—Let the st. line AB be divided into two parts at C;



Then shall
rects. $BA.AC$ and $AB.BC$ together = sq. on AB.

Construction—
1. On AB describe the square $ADEB$*...........I. 46.
2. Through C draw CF ||[l] to AD or BE..........I. 31.

Proof—

Sq. on AB = figure $ADEB$..........................Const.
= figs. AF and CE,
= rects. $DA.AC$ and $EB.BC$,
= rects. $BA.AC$ and $AB.BC$...........Const.
($\because$ $DA = EB = AB$)................Const.

Wherefore, *if a straight line, etc.*

Q.E.D.

Algebraic Proof.

Let $AC = a$, $CB = b$.
Then $AB = a + b$.

We have to prove that

$$(a+b)a + (a+b)b = (a+b)^2,$$
$$i.e.\ (a+b)(a+b) = (a+b)^2.$$

Q.E.D.

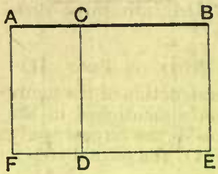
* See Note on page 115.

PROPOSITION III. THEOREM.

GEN. ENUN.—*If a straight line be divided into any two parts,*

Then the rectangle contained by the whole line and one of the parts *is equal to* *the rectangle contained by the two parts, together with the square on the aforesaid part.*

PART. ENUN.—Let the st. line AB be divided into two parts at C;



Then shall

rect. $AB \cdot BC$ = rect. $AC \cdot CB$ together with sq. on BC.

CONSTRUCTION—

1. On BC describe the square $BCDE$*...........I. 46.
2. Produce ED to F.
3. Through A draw AF ||[l] to CD or BE...........I. 31.

PROOF—

Rect. $AB \cdot BC$ = figure AE ($\because BE = BC$)............Const.
= figs. AD, and CE.
= rect. $AC \cdot CD$, and sq. on BC.........Const.
= rect. $AC \cdot CB$, and sq. on BC..($\because CD = CB$.)

WHEREFORE, *if a straight line, etc.*

Q.E.D.

ALGEBRAIC PROOF.

Let $AC = a$, $BC = b$.
Then $AB = a + b$.

We have to prove that

$(a+b)b = a.b + b^2$.

Which is evident. Q.E.D.

* See Note on page 115.

Exercises on Prop. I.

1. Prove that the Proposition is true when $BD = 2$ feet, $DE = 3$ feet, $EC = 5$ feet, $A = 4$ feet. Draw a figure to show the same result graphically. (Take a straight line about a quarter of an inch long to represent 1 foot.)

2. Prove that a room 8 feet wide and 160 square feet in area can be divided into three parts, each 8 feet long, and 5 feet, 7 feet, and 8 feet wide respectively. Draw a figure.

3. How do you know that DL is equal to the rectangle $A.DE$?

4. What modification of Ax. 9 is used in the second line of the Proof?

5. If one of the parts of BC be the same length as A, what does the corresponding rectangle become?

6. Suppose BC is divided into three *equal* parts; what will the Enunciation then become?

Note on Prop. II.

The first step in the construction of the figures in Props. II.—VIII. is always to describe a square mentioned in the Enunciation. If more than one square is mentioned, the largest is always described first.

Also, in Props. IV.—VIII. the second step is to draw the diagonal of the square.

Exercises on Prop. II.

1. Show that this is a particular case of Prop. I.

2. How do you know that CF can be drawn parallel to AD, *or* BE?

3. Prove the Proposition is true when AB is 8 inches and AC 5 inches.

4. How could the Enunciation be modified if C were the middle point of AB?

5. Give full reasons why $AD = BE = AB$. (Last line of Proof.)

6. What is the exact meaning of "$AC = a$," in the Algebraic Proof?

Exercises on Prop. III.

1. Prove that the rectangle $AB . AC$ is equal to the rectangle $AC . CB$ together with the square on AC.

2. What axiom is assumed at every line of this proof?

3. Prove the Proposition arithmetically, giving your own numbers for the lengths of the lines.

4. This Proposition is a special case of Prop. I. What corresponds here to x in the Algebraic Proof of Prop. I.?

5. What will the Enunciation become if AC is equal to BC?

Exercises on Prop. IV.

1. Prove that the complements AG, GE, are not only equal in area, but also in all respects.

2. Prove Corollary II.

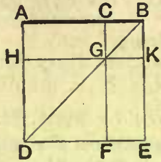
3. If, instead of the construction given, we took BK equal to BC, and drew KH parallel to AB, through K, and CF parallel to BE, through C, meeting KH in G; prove that BD would pass through G. (*I.e.*, join BG and GD, and prove them to be in the same straight line.)

PROPOSITION IV. THEOREM.

GEN. ENUN.—*If a straight line be divided into any two parts,*

$$\left.\begin{array}{l}\textit{Then the square on}\\ \textit{the whole line}\end{array}\right\}\begin{array}{c}\textit{is}\\ \textit{equal}\\ \textit{to}\end{array}\left\{\begin{array}{c}\textit{the squares on the two parts,}\\ \textit{together with twice the rectangle}\\ \textit{contained by the parts.}\end{array}\right.$$

PART. ENUN.—Let the straight line AB be divided into two parts at C;



Then shall

$$\textbf{sq. on } AB = \begin{cases}\textbf{sqs. on } AC \textbf{ and } CB,\\ \textbf{togr with twice the rect. } AC\,.\,CB.\end{cases}$$

CONSTRUCTION—

1. On AB describe the square $ADEB$..........I. 46.
2. Join BD.*
3. Through C draw CF $\|^l$ to AD or BE, cutting BD in G,...I. 31.
4. Through G draw HGK $\|^l$ to AB or DE........I. 31.

PROOF—

1. $\because$ BD meets the $\|^{ls}$ CF, AD.............Const.
$\therefore$ ext^r $\angle CGB$ = int^r opp^{te} $\angle ADB$............I. 29.
But $\angle ADB = \angle ABD$ ($\because AB = AD$).......I. 5.
$\therefore$ $\angle CGB = \angle CBG$..........................Ax. 1.
$\therefore$ $CG = CB$.............................I. 6.
But $CB = GK$ and $CG = BK$..........I. 34.
$\therefore$ *the figure CGKB is equilateral*...................Ax. 1.

2. $\because$ CB meets the $\|^{ls}$ CG, BK...............Const.
$\therefore$ the $\angle^s$ KBC, GCB = two rt $\angle^s$............I. 29.
But $\angle KBC$ is a rt. $\angle$...................Const.
$\therefore$ $\angle GCB$ is also a rt. $\angle$.
$\therefore$ the $\angle^s$ CGK, GKB opposite to them are also right angles ....I. 34.
$\therefore$ *the figure CGKB is rectangular.*
$\therefore$ *it is a square, and it is the square on BC*.......Def. 30.

* See Note on page 115.

3. Similarly we can show that HF is also a square,
And it is the square on HG, which $= AC$......I. 34.
$\therefore$ $HF =$ *square on* AC.

4. Sq. on $AB =$ figs. HF, CK, AG, GE.
$=$ sqs. on AC, BC, and twice AG......I. 43.
$=$ sqs. on AC, BC, and twice rect. $AC \,.\, CG$.
$=$ sqs. on AC, BC, and twice rect. $AC \,.\, CB$.
($\because$ $CG = CB$).........Proved above.

WHEREFORE, *if a straight line, etc.* Q.E.D.

COROLLARY I.—From this it is manifest, that parallelograms about the diameter of a square are likewise squares.

COROLLARY II.—The square on a straight line is equal to four times the square on half the line. (Take AC equal to CB.)

SECOND PROOF.

GEN. AND PART. ENUNS.—As above.

PROOF—Sq. on $AB =$ rects. $BA \,.\, AC$ and $AB \,.\, BC$...........II. 2.

$$= \left\{ \begin{array}{l} \text{rect. } AC \,.\, CB \text{ and sq. on } AC, \\ \text{with rect. } AC \,.\, CB \text{ and sq. on } BC \end{array} \right\} \text{...II. 3.}$$

$=$ sqs. on AC, BC, with twice rect. $AC \,.\, CB$.

WHEREFORE, *if a straight line, etc.* Q.E.D.

ALGEBRAIC PROOF.

Let $AC = a$, $BC = b$. Then $AB = a + b$,
We have to show that

$$(a + b)^2 = a^2 + b^2 + 2ab,$$

Which is easily proved by multiplication. Q.E.D.

NOTES.

This Proposition is one of the most important in Book II. The same figure is used either alone, or with additions, in Props V.—VIII., and Parts 1, 2, 3 of the Proof are not repeated in these Propositions, but taken as proved. The result of these three parts of the Proof is summed up in Corollary I.

In connexion with the Second Proof, it should be noticed that if CF only be drawn, we have the figure of Prop. II., (corresponding to line 1 of the Proof); and if then HK be drawn we have, above and below it, two figures of Prop. III. (lines 2 and 3 of the Proof).

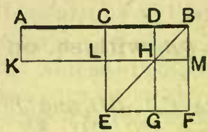
In this Second Method we lose the great advantage of actually seeing the equality of the areas: it, however, forms a useful exercise.

PROPOSITION V. THEOREM.

GEN. ENUN.—*If a straight line be divided into two equal and also into two unequal parts,*

Then the rectangle contained by the unequal parts, together with the square on the line between the points of section } *is equal to* { *the square on half the line.*

PART. ENUN.—Let the st. line AB be bisected at C, and divided unequally at D;



Then shall

rect. $AD \cdot DB$, with sq. on CD = sq. on CB.

CONSTRUCTION.—

1. On CB describe the sq. $CEFB$*.............I. 46.
2. Join BE.
3. Through D draw DHG $\|^l$ to CE or BF
4. Through H draw KLM $\|^l$ to CB or EF } ....I. 31.
5. Through A draw AK $\|^l$ to CL or BM

PROOF—

The rect. $AD \cdot DB$ with the sq. on CD } = { The fig. AH............($\because DH = DB$.) with the fig. LG ......II. 4. Cor. I.
= The figs. AL, CH, LG.
= The figs. AL, HF, LG,.........I. 43.
= The figs. CM, HF, LG...........I. 36.
= The figure CF.
= The square on CB.............Const.

WHEREFORE, *if a straight line, etc.*

Q.E.D.

* See Note on page 115.

PROPOSITION V. Theorem. (Second Proof.)

Gen. Enun.—*If a straight line be divided into two equal and also into two unequal parts,*

Then the rectangle contained by the unequal parts, together with the square on the line between the points of section *is equal to* *the square on half the line.*

Part. Enun.—Let the st. line AB be bisected at C, and divided unequally at D;

Then shall

Rect. $AD \,.\, DB$, with sq. on CD = sq. on CB.

Proof—

Sq. on CB = rects. $CB \,.\, BD$, and $BC \,.\, CD$ .... II. 2.

= { rect. $CB \,.\, BD$,
with rect. $CD \,.\, DB$ and sq. on CD...... II. 3.

= { rects. $AC \,.\, DB$, and $CD \,.\, DB$,
with sq. on CD...................... ($\because AC = CB$.)

= rect. $AD \,.\, DB$, with sq. on CD............. II. 1.

Wherefore, *if a straight line, etc.*

Q.E.D.

Algebraic Proof.

Let $AB = 2a$, $CD = b$,

Then $AC = a$, $BC = a$, $AD = a + b$, $BD = a - b$.

We have to show that

$$(a+b)(a-b) + b^2 = a^2$$

or, $$(a+b)(a-b) = a^2 - b^2,$$

a well-known identity. Q.E.D.

Exercises.

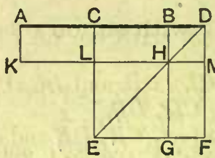
1. Prove by Playfair's Axiom that ML and AK will meet when produced.
2. Why do we not *prove* the fact that LG is equal to the square on CD?
3. Why would it be wrong in line 4 of the Construction to say "through K draw KLM parallel to CB or EF"?
4. What figure do CM and HF make up?
5. Prove the Proposition, when D lies between A and C.
6. Write out in full the proof that LG is equal to the square on CD.

PROPOSITION VI. THEOREM.

GEN. ENUN.—*If a straight line be bisected, and produced to any point,*

Then the rectangle contained by the whole line thus produced, and the part produced, together with the square on half the line bisected,	*is equal to*	*the square on the line made up of the half and the part produced.*

PART. ENUN.—Let the st. line AB be bisected at C, and produced to D;



Then shall

rect. $AD \cdot DB$, with sq. on CB = sq. on CD.

CONSTRUCTION—

1. On CD describe the sq. $CEFD$ .... I. 46.
2. Join DE.
3. Through B draw BHG ||^l to CE or DF
4. Through H draw KLM ||^l to AD or EF .... I. 31.
5. Through A draw AK ||^l to CL or DM

PROOF—

The rect. $AD \cdot DB$ with the sq. on CB = { The fig. AM .... ($\because DB = DM$). with the fig. LG ...... II. 4. Cor. I.

= The figs. AL, CH, BM, LG.

= The figs. AL, HF, BM, LG.. I. 43.

= The figs. CH, HF, BM, LG... I. 36.

= The figure CF.

= The square on CD .... Const.

WHEREFORE, *if a straight line, etc.*

Q.E.D.

PROPOSITION VI. THEOREM. (Second Proof.)

GEN. ENUN.—*If a straight line be bisected, and produced to any point,*

Then the rectangle contained by the whole line thus produced, and the part produced, together with the square on half the line bisected, } *is equal to* { *the square on the line made up of the half and the part produced.*

PART. ENUN.—Let the st. line AB be bisected at C, and produced to D;

Then shall

Rect. $AD \cdot DB$, **with sq. on** CB = **sq. on** CD.

PROOF—

Sq. on CD = rects. $CD \cdot DB$, and $DC \cdot CB$ ....II. 2.

$= \begin{cases} \text{rect. } CD \cdot DB, \\ \text{with rect. } CB \cdot DB \text{ and sq. on } CB \end{cases}$......II. 3.

$= \begin{cases} \text{rects. } CD \cdot DB, \text{ and } AC \cdot DB, \\ \text{with sq. on } CB, \end{cases}$......................($\because AC = CB$.)

= rect. $AD \cdot DB$, with sq. on CB,............II. 1.

WHEREFORE, *if a straight line, etc.* Q.E.D.

ALGEBRAIC PROOF.

Let $AB = 2a$, $CD = b$.

Then $AC = a$, $BC = a$, $AD = b + a$, $BD = b - a$.

We have to show that

$$(b+a)(b-a) + a^2 = b^2$$

or, $$(b+a)(b-a) = b^2 - a^2,$$

a well-known identity.

Q.E.D.

NOTE.

The Algebraic Proof for Props. V. and VI. shows that they are two cases of the same Theorem; viz., "*The rectangle contained by the sum and difference of two straight lines is equal to the difference of their squares.*"

In the second Proofs of Props. IV., V., and VI., we apply Prop. II. to the largest square mentioned in the Enunciation, and then Prop. III. to one or both of the rectangles thus obtained. (Cf. Note on Prop. IV.)

EXERCISES.

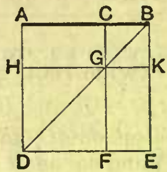
1. Prove geometrically that the rectangle contained by the sum and difference of two straight lines is equal to the difference of their squares.
2. Prove Prop. VI. Algebraically, taking $AB = 2a$, $BD = b$.

PROPOSITION VII. THEOREM.

GEN. ENUN.—*If a straight line be divided into any two parts,*

Then the squares on the whole line and on one of the parts } *are equal to* { *twice the rectangle contained by the whole and that part, together with the square on the other part.*

PART. ENUN.—Let the st. line AB be divided into two parts at C;



Then shall

the sqs. on AB, BC = $\left\{\begin{array}{l}\textbf{twice the rect. } AB.BC, \\ \textbf{together with the sq. on } AC.\end{array}\right.$

CONSTRUCTION—

The same as in Prop. IV.

PROOF—

The sqs. on AB, BC = The figs. AE, CK ....II. 4, Cor. I.

= The figs. AK, HF, GE, CK.

= The figs. AK, HF, CE.

= The figs. AK, HF, AK............Ax. 2.

(adding CK to the equals AG and GE)........I. 43.

$= \left\{\begin{array}{l}\text{Twice the fig. } AK, \\ \text{with the fig. } HF\end{array}\right.$

$= \left\{\begin{array}{l}\text{Twice the rect. } AB.BC, (\because BC = BK.) \\ \text{with sq. on } AC\text{............II. 4. Cor. I.}\end{array}\right.$

WHEREFORE, *if a straight line, etc.*

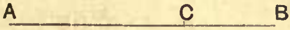
Q.E.D.

PROPOSITION VII. THEOREM. (Second Proof.)

GEN. ENUN.—*If a straight line be divided into any two parts,*

Then the squares on the whole line and on one of the parts are equal to *twice the rectangle contained by the whole and that part, together with the square on the other part.*

PART. ENUN.—Let the st. line AB be divided into two parts at C;



Then shall
sqs. on AB, BC = twice rect. $AB.BC$, with sq. on AC.

PROOF—

Sq. on AB, and sq. on BC = sqs. on AC, CB, with twice rect. $AC.CB$, and sq. on BC .... II. 4.

= twice sq. on BC, twice rect. $AC.CB$, and sq. on AC.

= twice rect. $AB.BC$* and sq. on AC ... II. 3.

WHEREFORE, *if a straight line, etc.*

Q.E.D.

ALGEBRAIC PROOF.

Let $AB = a$, $BC = b$,
Then $AC = a - b$.
We have to prove that
$$a^2 + b^2 = 2ab + (a-b)^2,$$
or, $a^2 - 2ab + b^2 = (a-b)^2$:
Which is easily proved by Multiplication.

Q.E.D.

EXERCISES.

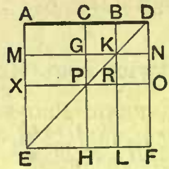
1. Prove that AK is equal to CE in all respects.
2. Which part of the figure is taken twice over in this Enunciation?
3. Prove that AF is equal to HE.
4. Write out the Construction for this Proposition in full.

* Rect. $AB.BC$ = sq. on BC, and rect. $AC.CB$,
∴ Twice rect. $AB.BC$ = twice sq. on BC, and twice rect. $AC.CB$.

PROPOSITION VIII. Theorem. (First Proof.)

Gen. Enun.—*If a straight line be divided into any two parts, four times the rectangle contained by the whole line and one of the parts, together with the square on the other part* is equal to *the square on the straight line made up of the whole and the first part.*

Part. Enun.—Let the st. line AB be divided into two parts at C;



Then shall

Four times the rect. $AB \cdot BC$ together with the sq. on AC = the sq. on the line made up of AB, BC.

Construction—

1. Produce AB to D, so that $BD = BC$............I. 3.
2. On AD describe the square $AEFD$............I. 46.
3. Construct two figures such as in the preceding propositions.

Proof—

1. $\because CB = GK = PR$........................I. 34.
and $BD = KN = RO$........................I. 34.
and $CB = BD$........................Const.
$\therefore CB, BD, GK, KN, PR, RO$, are all equal.....Ax. 1.
Similarly, DN, NO, BK, KR, CG, GP are all equal.
Hence, the rects. CK, BN, GR, KO, are all equal..I. 36.

And $\because CG = GP$,....................Just proved.
$\therefore$ rect. AG = rect. MP.......... ....I. 36.
Similarly, rect. PL = rect. RF.
But rect. MP = rect. PL.....................I. 43.
$\therefore$ rects. AG, MP, PL, RF, are all equal.....Ax. 1.

2. $\left.\begin{array}{l}\text{4 times rect. } AB \,.\, BC \\ \text{with sq. on } AC\end{array}\right\} = \left\{\begin{array}{l}\text{4 times } AK \text{.. } (\because BC = BK) \\ \text{with fig. } XH \text{...II.4, Cor. I.}\end{array}\right.$

$= \left\{\begin{array}{l}\text{4 times } AG\text{, 4 times } CK\text{,....II. 1.} \\ \text{and the fig. } XH.\end{array}\right.$

$= \left\{\begin{array}{l}\text{the figs. } AG, MP, PL, RF, \\ CK, BN, GR, KO, \text{and } XH \text{.. Proof (1).}\end{array}\right.$

$=$ the figure AF,

$=$ the square on AD,...............Const,

$= \left\{\begin{array}{l}\text{the square on the line} \\ \text{made up of } AB \text{ and } BC \text{......Const.}\end{array}\right.$

WHEREFORE, *if a straight line, etc.*

Q.E.D.

ALGEBRAIC PROOF.

Let $AB = a$, $BC = b$,

Then $AC = a - b$, $AD = a + b$.

We have to show that

$$4ab + (a-b)^2 = (a+b)^2,$$

i.e., $$(a+b)^2 - (a-b)^2 = 4ab,$$

i.e., $$a^2 + 2ab + b^2 - (a^2 - 2ab + b^2) = 4ab.$$

Which is evident.

Q.E.D.

EXERCISES.

1. Prove that AG is equal to RF in all respects.
2. Which figure is equal to the square on AB? Prove the truth of your answer.
3. Which are the two figures spoken of in part 3 of the Construction?
4. Show that this Proposition incidentally affords a proof of II. 4, Cor. II.
5. If AC is equal to CD, prove that the square on AD is 16 times the square on BC.
6. Prove that four times the rectangle $AB \,.\, AC$, together with the square on BC, is equal to the square on the line made up of AB, AC.
7. Prove this Proposition Algebraically when $AC = a$, $CB = b$.
8. Show from the Algebraic Proof that this Proposition may be regarded as a special case of the Theorem enunciated in the Note to Prop. VI.

PROPOSITION VIII. THEOREM. (Second Proof.)

GEN. ENUN.—*If a straight line be divided into any two parts, four times the rectangle contained by the whole and one of the parts, together with the square on the other part, is equal to the square on the straight line made up of the whole and the first part.*

PART. ENUN.—Let the st. line AB be divided into two parts at C;

Then shall

Four times the rect. $AB \cdot BC$ together with the sq. on AC = the sq. on the line made up of AB, BC.

CONSTRUCTION—

Produce AB both ways to D and E,
Making $BD = BC$ and $AE = AC$..............I. 3.

PROOF—

$\because$ EC is bisected at A, and produced to D......Const.
$\therefore$ the rect. $ED \cdot DC$ with sq. on AC = sq. on AD...II. 6.
and ED = twice AB.......................Const.
$\therefore$ twice the rect. $AB \cdot DC$ with sq. on AC = sq. on AD...II. 1.
and DC = twice BC.......................Const.
$\therefore$ four times the rect. $AB \cdot BC$ with sq. on AC = sq. on AD.......II. 1.
= sq. on line made up of AB and BC.......Const.

WHEREFORE, *if a straight line, etc.*

Q.E.D.

ALGEBRAIC PROOF.

Let $AB = a$, $BC = b$,
Then $AC = a - b$, $AD = a + b$.

We have to show that

$$4ab + (a-b)^2 = (a+b)^2,$$

i.e., $$(a+b)^2 - (a-b)^2 = 4ab,$$

i.e., $$a^2 + 2ab + b^2 - (a^2 - 2ab + b^2) = 4ab.$$

Which is evident. Q.E.D.

Exercises. See page 125.

NOTES ON PROPS. IV.–VIII.

We see from the Algebraic Proofs of Propositions IV.—VII., that they are the Geometrical properties corresponding to the three important Algebraic formulæ :—

$$(a+b)^2 = a^2 + 2ab + b^2 \qquad \text{(II. 4).}$$
$$(a+b)(a-b) = a^2 - b^2 \qquad \text{(II. 5, 6).}$$
$$(a-b)^2 = a^2 - 2ab + b^2 \qquad \text{(II. 7).}$$

Prop. VIII. may be regarded as the result of combining Props. IV. and VII. (subtracting equals from equals); or, if we take $AD = 2a$, and $AC = 2b$, we shall find the resulting Algebraic formula is

$$(a+b)(a-b) = a^2 - b^2$$

The Explanation of the single result obtained from Props. V. and VI. may be stated thus :—

Fig. of Prop. V.

Fig. of Prop. VI.

In both cases AB is bisected, but in Prop. V. it is unequally divided into two parts *internally*, and in Prop. VI. it may be considered as divided into two parts (AD and DB) *externally*. In Prop. V., passing from A to B through D, we proceed always in the same direction. But in Prop. VI., after going from A to D, we must go *back* to B; and this change of direction corresponds to a negative sign in Algebra, as may be shown thus :—

If a quantity a be divided into two parts, one of which is x, then the other will be $a - x$.

Now suppose x is greater than a,

i.e., that $x = a + b$,

Then $a - x = a - (a + b) = -b$,

i.e., the other part of a is a negative quantity.

Calling $AB = a$, and $AD = x$, in Props. V. and VI., we easily see that change of direction in Geometry, corresponds to a negative sign in Algebra.

If we take $AC = a$, and $CD = b$, as in the Algebraic Proof given above, we see that Prop. V. is for the case $a > b$, and Prop. VI. for the case $b > a$.

EXERCISES.

1. Explain in your own words, how it is that Props. V. and VI. are cases of the same Proposition, and what is the difference between them.

2. Remembering the Note above on direction of lines, show that in the figure of Prop. VI.,

$$AB + BD + DA = 0.$$

3. If a man has to go from A to D (Figure of Prop. V.), and goes by mistake to B first, show that this gives an explanation of the equation

$$AB + BD = AD.$$

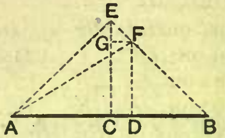
Compare with figure of Prop. VI.

PROPOSITION IX. THEOREM. (First Proof.)

GEN. ENUN.—*If a straight line be divided into two equal and also into two unequal parts,*

Then the squares on the two unequal parts } *are together double of* { *the square on half the line, and the square on the line between the points of section.*

PART. ENUN.—Let the st. line AB be divided into two equal parts at C, and into two unequal parts at D;



Then shall
sqs. on AD, DB = twice the sqs. on AC, CD.

CONSTRUCTION—

1. From C draw CE $\perp^r$ to AB .... I. 11.
and make it equal to AC or CB .... I. 3.
2. Join EA, EB.
3. Through D draw DF $\|^l$ to CE, meeting EB in F,
and through F draw FG $\|^l$ to AB .... I. 31.
4. Join AF.

PROOF—

1. $\because$ $AC = CE$ .... Const.
$\therefore$ $\angle EAC = \angle AEC$ .... I. 5.
and $\because$ $\angle ACE$ is a rt. $\angle$ .... Const.
$\therefore$ $\angle^s$ EAC and AEC tog^r = a rt. $\angle$ .... I. 32.
and they are equal to one another,
$\therefore$ Each is half a rt. $\angle$.

Similarly each of the $\angle^s$ CEB, EBC is half a rt. $\angle$,
$\therefore$ the whole $\angle AEB$ is a rt. $\angle$.

2. $\because$ $\angle GEF$ is half a rt. $\angle$............Just proved.
and $\angle EGF$ is a rt. $\angle$, ($\because$ $\angle EGF = \angle ECB$)..........I. 29.
$\therefore$ rem^{g} EFG is half. a rt. $\angle$................I. 32.
$\therefore$ $\angle GEF = \angle EFG$,
$\therefore$ $EG = GF$..........................I. 6.

3. $\because$ $\angle FBD$ is half a rt. $\angle$............Proved above.
and $\angle FDB$ is a rt. $\angle$, ($\because$ $\angle FDB = \angle ECB$)..........I. 29.
$\therefore$ rem^{g} $\angle BFD$ is half a rt. $\angle$................I. 32.
$\therefore$ $\angle FBD = \angle BFD$,
$\therefore$ $DF = DB$..........................I. 6.

4. The sqs. on AD, DB

$$
\begin{aligned}
&= \text{sqs. on } AD, DF && (\because DB = DF) \text{..Proved above.} \\
&= \text{sq. on } AF && (\because \angle ADF \text{ is a rt. } \angle) \\
&= \text{sqs. on } AE, EF && (\because \angle AEF \text{ is a rt. } \angle) \quad \text{I. 47.} \\
&= \begin{cases} \text{sqs. on } AC, CE & (\because ACE \text{ is a rt. } \angle) \\ \text{with sqs. on } EG, GF & (\because EGF \text{ is rt. } \angle) \end{cases} \\
&= \begin{cases} \text{twice sq. on } AC & (\because AC = CE). \\ \text{with twice sq. on } GF & (\because EG = GF). \end{cases} \\
&= \text{twice sqs. on } AC, CD && (\because CD = GF) \text{.....I. 34.}
\end{aligned}
$$

WHEREFORE, *if a straight line, etc.*

Q.E.D.

ALGEBRAIC PROOF.

Let $AB = 2a$, $CD = b$,
Then $AC = a$, $BC = a$, $AD = a + b$, $BD = a - b$.
We have to show that

$$(a+b)^2 + (a-b)^2 = 2a^2 + 2b^2,$$

i.e. $a^2 + 2ab + b^2 + a^2 - 2ab + b^2 = 2a^2 + 2b^2$.

Which is evident.

Q.E.D.

EXERCISES.

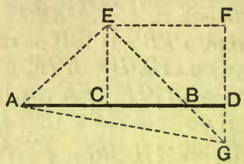
1. Why is GF equal to CD?
2. Prove that the angle AFB is obtuse.
3. How do you know that the angle EGF is equal to the angle GCD, and the angle FDB to the angle ECB?
4. Explain fully why the squares on AE, EF, are equal to the squares on AC, CE, EG, GF. (Proof 4.)

PROPOSITION X. THEOREM. (First Proof.)

GEN. ENUN.—*If a straight line be bisected and produced to any point,*

Then the square on the whole line thus produced, and the square on the part of it produced } *are together double of* { *the square on half the line bisected, and the square on the line made up of the half and the part produced.*

PART. ENUN.—Let the st. line AB be bisected at C, and produced to D;



Then shall
the sqs. on AD, DB = twice the sqs. on AC, CD.

CONSTRUCTION—

1. From C draw $CE \perp^r$ to AB ....I. 11.
   and make it equal to AC or CB.... ....I. 3.
2. Join AE, EB.
3. Through E draw $EF \parallel^l$ to AB,
   and through D draw $DF \parallel^l$ to CE.........I. 31.

SUBSIDIARY PROOF. {
$\because$ EF meets the $\parallel^{ls}$ EC, FD,
$\therefore$ $\angle^s$ CEF, EFD = two rt. $\angle^s$...............I. 29.
$\therefore$ $\angle^s$ BEF, EFD are less than two rt. $\angle^s$,.......Ax. 9.
$\therefore$ lines EB, FD will meet, if produced
towards B, D...............................Ax. 12.

4. Let them be produced and meet in G.
5. Join AG.

PROOF—

1. $\because$ $AC = CE$........................Const.
   $\therefore$ $\angle EAC = \angle AEC$.......................I. 5.
   and $\because$ $\angle ACE$ is a rt. $\angle$..................Const.
   $\therefore$ $\angle^s$ EAC and AEC tog^r = a rt. $\angle$............I. 32.
   and they are equal to one another,
   $\therefore$ each is half a rt. $\angle$.

Similarly each of the $\angle^s$ CEB, EBC is half a rt. $\angle$,
$\therefore$ the whole $\angle AEB$ is a rt. $\angle$.

2. $\because \angle EBC$ is half a rt. $\angle$............Just proved.
$\therefore \angle DBG$ is half a rt. $\angle$..................I. 15.
and BDG is a rt. $\angle$, ($\because \angle BDG = \angle DCE$)............I. 29.
$\therefore$ rem^g $\angle DGB$ is half a rt. $\angle$...............I. 32.
$\therefore \angle DGB = \angle DBG$,
$\therefore BD = DG$..........................I. 6.

3. $\because \angle EGF$ is half a rt. $\angle$............Just proved.
and $\angle EFG$ is a rt. $\angle$, ($\because \angle EFG = \angle ECD$)............I. 34.
$\therefore$ rem^g $\angle FEG$ is half. a rt. $\angle$...............I. 32.
$\therefore \angle FEG = \angle EGF$,
$\therefore GF = FE$..........................I. 6.

4. Sqs. on AD, DB

$$
\begin{aligned}
&= \text{sqs. on } AD, DG \quad (\because DG = DB) \text{.....Proof (2).}\\
&= \text{sq. on } AG \quad (\because \angle ADG \text{ is a rt. } \angle)\\
&= \text{sqs. on } AE, EG \quad (\because \angle AEG \text{ is a rt. } \angle)\\
&= \left\{\begin{array}{l}\text{sqs. on } AC, CE \quad (\because \angle ACE \text{ is a rt. } \angle)\\ \text{and sqs. on } EF, FG \; (\because \angle EFG \text{ is a rt. } \angle)\end{array}\right\} \text{I. 47.}\\
&= \left\{\begin{array}{l}\text{twice sq. on } AC \quad (\because AC = CE) \text{.......Const.}\\ \text{with twice sq. on } EF \; (\because EF = FG) \text{..Proof (3).}\end{array}\right.\\
&= \text{twice sqs. on } AC, CD \; (\because EF = CD) \text{.......I. 34.}
\end{aligned}
$$

WHEREFORE, *if a straight line, etc.*

Q.E.D.

N.B.—If Playfair's Axiom be used, the Subsidiary Proof is as follows:—

$\because EC$ is $\|^l$ to FD, $\therefore EB$ cannot be $\|^l$ to FD...Playfair's Ax.
$\therefore EB$ and FD will meet if produced.

ALGEBRAIC PROOF.

Let $AB = 2a$, $CD = b$,
Then $AC = a$, $BC = a$, $AD = b + a$, $BD = b - a$.
We have to show that

$$(b + a)^2 + (b - a)^2 = 2b^2 + 2a^2,$$

i.e., $b^2 + 2ab + a^2 + b^2 - 2ab + a^2 = 2b^2 + 2a^2$.
Which is evident.

Q.E.D.

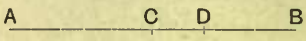
PROPOSITIONS IX. AND X. THEOREMS. (Second Proof.)

IX.

GEN. ENUN.—*If a straight line be divided into two equal, and also into two unequal parts,*

Then the squares on the two unequal parts are together double of the square on half the line, and the square on the line between the points of section.

PART. ENUN.—Let the st. line AB be divided into two equal parts at C, and into two unequal parts at D;



Then shall

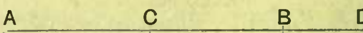
sqs. on AD, DB = twice the sqs. on AC, CD.

X.

GEN. ENUN.—*If a straight line be bisected, and produced to any point,*

Then the square on the whole line thus produced, and the square on the part of it produced are together double of the square on half the line bisected, and the square on the line made up of the half and the part produced.

PART. ENUN.—Let the st. line AB be bisected at C, and produced to D.



Then shall

the sqs. on AD, DB = twice the sqs. on AC, CD.

PROOF—

(of both Props.) 1. Sq. on AD = { sqs. on AC, CD, with twice rect. $AC \cdot CD$ ....II. 4.

= { sqs. on AC, CD, with twice rect. $BC \cdot CD$ ..($\because AC = CB$).

2. twice rect. $BC \cdot CD$, with sq. on BD } = sqs. on BC, CD ....II. 7.

$\therefore$ adding equals to equals,

$$\left.\begin{array}{l}\text{the sqs. on } AD, DB\\ \text{with twice the rect.}\\ BC.CD\end{array}\right\} = \left\{\begin{array}{l}\text{sqs. on } AC, BC, \text{ twice}\\ \text{sq. on } CD, \text{ and twice}\\ \text{rect. } BC.CD \text{..............Ax. 2}\end{array}\right.$$

3. Take away twice the rect. $BC.CD$.

$$\therefore \text{ the sqs. on } AD, DB = \left\{\begin{array}{l}\text{sqs. on } AC, BC, \text{ with}\\ \quad\text{twice sq. on } CD \text{........Ax. 3.}\end{array}\right.$$

$\therefore$ the sqs. on AD, DB = twice sqs. on AC, CD..($\because AC=BC$).

WHEREFORE, *if a straight line, etc.*

Q.E.D.

NOTE.

First, "II. 4" is applied to AD, divided at C.

Then, "II. 7" (reversed) is applied to the part which remains after taking away AC.

The form of the Algebraic Proof shows that these two Propositions are different cases of the same Theorem.

If II. 7 be written in direct, instead of reverse, order, the above will give us another Proof of Prop. VIII.

The following is a Summary of the Props. used in the Second Proofs of these Propositions :—

In Prop. IV..........2, 3.	In Prop. VII..........4, 3.
,, V..........2, 3, 1.	,, VIII..........6, 1, 1.
,, VI..........2, 3, 1.	,, IX., X..........4, 7.

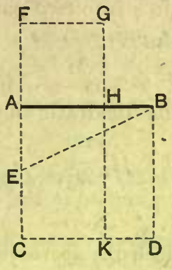
EXERCISES.

1. Show how Props. IX. and X. may be regarded as two cases of the same Theorem, in a similar way to Note on Props. V. and VI.
2. What relation do these Propositions bear to Prop. VIII. ?
3. In the figure of II. 10 (First Proof), prove that GF is equal to CD.
4. If a straight line AB be produced to C, show that the squares on AB and AC = twice the rectangle $AB.AC$ together with the square on BC.
5. Show by an Algebraic Proof of the above that it is a special case of Prop. VII., and indicate how it may be considered so, Geometrically.

PROPOSITION XI. PROBLEM.

GEN. ENUN.—*To divide a straight line into two parts, so that the rectangle contained by the whole and one of the parts may be equal to the square on the other part.*

PART. ENUN.—Let AB be the given st. line;



It is required to divide it into two parts, (at H, say) so that rect. $AB.BH$ may = sq. on AH.

CONSTRUCTION—

1. On AB describe the sq. $ACDB$, ...I. 46.
2. Bisect AC at E, and join BE,..............I. 10.
3. Produce EA to F, so that $EF = EB$............I. 3.
4. On AF describe the square $AFGH$............I. 46.
5. Produce GH to meet CD in K.

Then shall AB be divided at H, so that rect. $AB.BH$ = sq. on AH.

PROOF—

$\because$ AC is bisected at E, and produced to F.

$\therefore$ rect. $CF.FA$, with sq. on AE = sq. on EF.........II. 6.
= sq. on EB........Const.
= sqs. on EA, AB,..I. 47.

Take away from both the sq. on AE.

$\therefore$ rect. $CF.FA$ = sq. on AB..............Ax. 3.
i.e., fig. FK = fig. AD.........($\because$ $FG = FA$).

Take away from both the fig. AK.

$\therefore$ fig. $FH =$ fig. HD....................Ax. 3.

i.e., sq. on $AH =$ rect. $AB \cdot BH$.....($\because AB = BD$).

WHEREFORE, *the straight line AB has been divided, etc.*

Q.E.F.

ALGEBRAIC SOLUTION.

Let $AB = a$, $AH = x$.

It is required to divide a into two parts, x and $a - x$, so that

$$a(a - x) = x^2,$$

$$i.e., \quad a^2 - ax = x^2,$$

$$\text{or} \quad x^2 + ax - a^2 = 0,$$

which is the quadratic to determine x.

Solving it, we find $AH = a\dfrac{\sqrt{5} - 1}{2}$, $BH = a\dfrac{3 - \sqrt{5}}{2}$.

NOTES.

The second solution which the above quadratic equation also gives, viz., $AH = -a\dfrac{\sqrt{5}+1}{2}$, $BH = a\dfrac{\sqrt{5}+3}{2}$, corresponds to the Problem:—

"*Produce a straight line, so that the rectangle contained by the whole,*
"*and the line made up of the whole and the part produced, may be*
"*equal to the square on the part produced.*"

The positive and negative signs of this second solution point to the geometrical fact that the straight line AB is to be divided *externally*. Cf. Note on Prop. V. and VI.

The Construction for this Problem is similar to that of Prop. XI.

1, 2, 3. As in Proposition.
4. On CF describe the square $CFMN$.
5. Produce BA to meet MN in L.

Then shall rect. $AB \cdot BL =$ sq. on AL.

The proof is left as an exercise for the student.

EXERCISES.

1. Prove that AF is greater than AE.

2. After describing the square $AFGH$, how do you know that its point H will lie on AB?

3. If FG, DB be produced to meet in X, and CH, HX joined, what conclusion would you draw about CH, HX, supposing the converse of I. 43 to be true?

4. Write out in full the solution and proof of the Problem given in the above Notes.

PROPOSITION XII. Theorem.

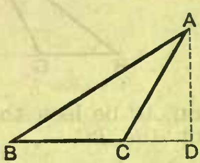
Gen. Enun.—*In obtuse-angled triangles,*

if a perpendicular be drawn from either of the acute angles to the opposite side produced,

$$\left.\begin{array}{l}\textit{Then the square on the side}\\ \textit{subtending the obtuse angle}\end{array}\right\}\begin{array}{c}\textit{is}\\ \textit{greater}\\ \textit{than}\end{array}\left\{\begin{array}{l}\textit{the squares on the sides}\\ \textit{containing the obtuse angle}\end{array}\right.$$

by *twice the rectangle contained by the side on which, when produced, the perpendicular falls, and the straight line intercepted without the triangle, between the perpendicular and the obtuse angle.*

Part. Enun.—Let ABC be a $\triangle$, obtuse-angled at C, and from A let AD be drawn $\perp^r$ to BC produced;



Then shall

sq. on BA be greater than sqs. on AC, CB, by twice rect. $DC \,.\, CB$.

Proof—

$$\text{Sq. on } BA = \left\{\begin{array}{l}\text{sq. on } BD,\\ \text{and sq. on } DA \ldots\ldots\ldots\ldots\ldots\ldots \text{I. 47.}\end{array}\right.$$

$$= \left\{\begin{array}{l}\text{sqs. on } BC, CD, \text{with twice rect. } DC \,.\, CB \text{ ..II. 4.}\\ \text{and sq. on } DA.\end{array}\right.$$

$$= \left\{\begin{array}{l}\text{sqs. on } CD,\ DA,\\ \text{with sq. on } BC, \text{and twice rect. } DC \,.\, CB.\end{array}\right.$$

$$= \left\{\begin{array}{l}\text{sq. on } AC \ldots\ldots\ldots\ldots\ldots\ldots\ldots \text{I. 47.}\\ \text{and sq. on } BC \text{ with twice rect. } DC \,.\, CB.\end{array}\right.$$

$\therefore$ sq. on BA *is greater than* sqs. on AC, CB by twice the rect. $DC \,.\, CB$.

Wherefore, *in obtuse-angled triangles, etc.*

Q.E.D.

Exercise.

State and prove the converse of this Proposition by an *indirect* proof after the manner of I. 19, and I. 25.

PROPOSITION XIII. THEOREM.

GEN. ENUN.—*In every triangle,*
the square on the side subtending an acute angle is less than *the squares on the sides containing that angle*
by *twice the rectangle contained by either of these sides, and the straight line intercepted between the perpendicular let fall on it from the opposite angle, and the acute angle.*

PART. ENUN.—Let ABC be a $\triangle$, acute-angled at B, and on BC let fall the $\perp^r$ AD from the opp^{te} $\angle$ A.

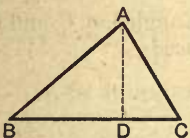


FIG. 1.

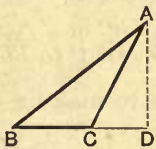


FIG. 2.

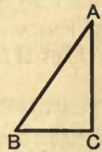


FIG. 3.

Then shall sq. on AC be less than sqs. on AB, BC by twice rect. $DB \cdot BC$.

PROOF—$\because$ BC is divided at D (Fig. 1), or BD at C (Fig. 2),

$$\therefore \text{ sqs. on } DB, BC = \begin{cases} \text{twice rect. } DB \cdot BC \\ \text{and sq. on } CD \end{cases} \quad \text{.............II. 7.}$$

Add to each the sq. on DA,

$$\therefore \left.\begin{array}{l} \text{sqs. on } BD, DA \\ \text{and sq. on } BC \end{array}\right\} = \begin{cases} \text{twice rect. } DB \cdot BC \\ \text{and sqs. on } CD, DA \end{cases} \quad \text{....Ax. 2.}$$

$$i.e. \left.\begin{array}{l} \text{sq. on } AB, \\ \text{and sq. on } BC \end{array}\right\} = \begin{cases} \text{twice rect. } DB \cdot BC, \\ \text{and sq. on } CA \end{cases} \quad \text{............I. 47.}$$

$\therefore$ sq. on AC *is less than* sqs. on AB, BC, by twice the rect $DB \cdot BC$.

In Fig 3, BC is the st. line between the $\perp^r$ and acute $\angle$,
$\therefore$ the rectangle in this case is the sq. on BC.

Now sq. on AB = sqs. on AC, CB..............I. 47.

Add to each the sq. on BC,

$$\therefore \text{ sqs. on } AB, BC = \begin{cases} \text{sq. on } AC \text{ with} \\ \text{twice sq. on } BC \end{cases} \quad \text{..........Ax. 2.}$$

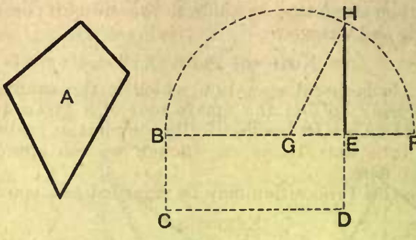
$\therefore$ sq. on AC *is less than* sqs. on AB, BC by twice the square on BC.

WHEREFORE, *in every triangle, etc.* Q.E.D.

PROPOSITION XIV. PROBLEM.

GEN. ENUN.—*To describe a square that shall be equal to a given rectilineal figure.*

PART. ENUN.—Let A be the given rectilineal figure;



It is required to make a square equal to A.

CONSTRUCTION—

1. Describe the rt.-angled $\square^{\text{gram}}BCDE$, equal to A ...I. 45.

Then, if $BE=ED$, the figure is a square, and what is required is done.

But if not,

2. Produce BE to F, so that $EF=ED$............I. 3.
3. Bisect BF at G.........................I. 10.
4. From cr. G, at dist^ce^ GF, describe the semi-$\odot^{\text{le}}$ BHF.
5. Produce DE to meet the semi-circle in H.
6. Join GH.

Then shall the sq. on EH=the figure A.

PROOF—

$\because$ BF is divided equally at G, and unequally at E,

$\therefore$ the rect. $BE.EF$, with sq. on GE $\}$ = sq. on GF.....................II. 5.

= sq. on GH................I. Def. 15.

= sqs. on GE, EH..............I. 47.

Take away from both the sq. on GE,

$\therefore$ the rect. $BE.EF$ = sq. on EH.............Ax. 3.

But the rect. $BE.EF$ = figure A................Const.

$\therefore$ the figure A = sq. on EH.............Ax. 1.

WHEREFORE, *the side of the square equal to the given figure, has been found.* Q.E.F.

Notes on Props. XII., XIII.

Prop. XII. refers only to obtuse-angled triangles.

Prop. XIII. to every triangle, whether acute-angled (Fig. 1), obtuse-angled (Fig. 2), or right-angled (Fig. 3).

To remember which is the rectangle, notice that in both propositions, in naming it, we go from the foot of the perpendicular to *the vertex of the angle*, and then along the same side to the end of it; or, more shortly, "into the angle and out again."

Note on Prop. XIV.

This is the Geometrical operation which is the counterpart of the Algebraic Problem "to find the square root of a given quantity." It may be noticed that whilst Geometrically the solution is always possible, Algebraically it is not always so, though we can approximate to it as closely as we wish.

In this sense this Proposition may be regarded as a converse of I. 46.

Exercises.

1. Write out the Proposition, producing CB to K, so that $BK = BE$ instead of producing BE.
2. Of what shape must A be?
3. Construct the figure *accurately*, making $BD = A$ by the method of I. 45.

SUMMARY

Of the Algebraic forms of the Propositions in Book II.

PROP.	ALGEBRAIC FORM.
I.	$x(a+b+c+\ldots\ldots) = ax+bx+cx+\ldots\ldots$
II.	$(a+b)a+(a+b)b = (a+b)^2.$
III.	$(a+b)b = ab+b^2.$
IV.	$(a+b)^2 = a^2+2ab+b^2.$
V., VI.	$(a+b)(a-b) = a^2-b^2.$
VII.	$(a-b)^2 = a^2-2ab+b^2.$
VIII.	$(a+b)^2-(a-b)^2 = 4ab.$
IX., X.	$(a+b)^2+(a-b)^2 = 2a^2+2b^2.$
XI.	Solve the equation, $x^2+ax-a^2=0.$
XII.	———
XIII.	———
XIV.	Find the square root of a.

Propositions XII. and XIII. are closely connected with the important Trigonometrical formula

$$a^2 = b^2 + c^2 - 2bc \cos A.$$

It will thus be seen that Book II. exhausts all the simple combinations of the Second Degree which can be formed with two letters.

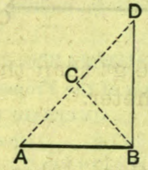
If the connexions between the Geometrical and Algebraic Problems are carefully studied, the above list will form a very easy means of remembering which is which, of the Propositions of Book II.; the enunciations of which often bewilder a beginner on account of their great verbal similarity.

APPENDIX.

I. Problem.

To draw a straight line perpendicular to another straight line from one end of it, without producing the given line.

Let AB be the given straight line and B the given end;



It is required to draw from B, a st. line $\perp^r$ to AB.

Construction—

On AB describe an isosceles $\triangle ABC$.

Produce AC to D, making $CD = CA$, and join BD.

BD shall be $\perp^r$ to AB.

Proof—

$$\left.\begin{array}{r} \because\quad \angle CAB = \angle CBA \\ \text{and } \angle CBD = \angle CDB \end{array}\right\} \text{..................I. 5.}$$

$\therefore\quad \angle ABD = \angle^s\ BAD$ and BDA............Ax. 2.

But these three $\angle^s =$ two rt. $\angle^s$...............I. 32.

$\therefore\quad \angle ABD = 1$ rt. $\angle$.

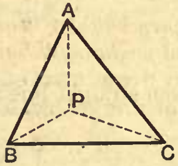
Wherefore, *a straight line has been drawn, etc.*

Q.E.F.

II. Theorem.

If from a point within a triangle, straight lines be drawn to the angular points of the triangle, these three straight lines shall be greater than half the perimeter of the triangle, but less than the whole perimeter.

Let ABC be the $\triangle$, and P the given pt.;



PA, PB, PC **shall be gr than the half, and less than the whole, perimeter.**

Proof—

1. $$\because \left.\begin{array}{llll} PB, PC & \text{are g}^{\text{r}} & \text{than} & BC \\ PC, PA & ,, & ,, & AC \\ PA, PB & ,, & ,, & AB \end{array}\right\} \text{..................I. 20.}$$

$\therefore$ twice PA, PB, PC are greater than AB, BC, CA.

$\therefore$ PA, PB, PC are greater than half the perimeter.

2. $$\because \left.\begin{array}{llll} BP, PC & \text{are less} & \text{than} & BA, AC \\ CP, PA & ,, & ,, & CB, BA \\ AP, PB & ,, & ,, & AC, CB \end{array}\right\} \text{...........I. 21.}$$

$\therefore$ twice PA, PB, PC are less than twice AB, BC, CA.

$\therefore$ PA, PB, PC are less than the perimeter.

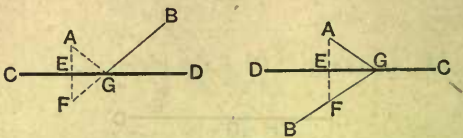
Wherefore, *if from a point, etc.*

Q.E.D.

III. Problem.

From two given points, to draw two straight lines which shall make equal angles with a given straight line.

Let A, B, be the given pts., and CD the given st. line;



It is required to draw two st. lines from A and B, making equal angles with CD.

Construction—

From A, draw $AE \perp^r$ to CD................I. 12.
Produce AE to F, so that $EF = AE$............I. 3.
Join FB, and let it (produced if necessary) meet CD in G.
Join AG.

AG, BG, **shall be the lines required.**

Proof—

In the $\triangle^s$ AEG, FEG

$$\because \begin{cases} AE = FE \text{ ....Const.} \\ EG = EG, \\ \text{and } \angle AEG = \angle FEG \text{ ....Const.} \end{cases}$$

$\therefore \quad \angle AGE = \angle EGF$........................I. 4.
$= \angle BGD$.....................I. 15.

Wherefore, *AG and GB are drawn as required.*

Q.E.F.

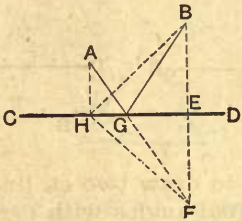
Note.

The above Proposition is of use in questions dealing with the reflexion of light.

IV. Problem.

In a given straight line to find a point such that the sum of its distances from two given points on the same side of the line shall be a minimum (i.e. the least possible).

Let A, B be the points, CD the given st. line ;



It is required to find a point (G) **in** CD **such that** AG, GB **shall be a minimum.**

Construction—

From A and B draw AG, BG making equal angles with CD....................................App. iii.

G **shall be the point required.**

Take any other pt. H, in CD, and join AH, BH, FH.

Proof—

In the $\triangle^s$ BEG, FEG

$$\because \left\{ \begin{array}{c} BE = FE \\ EG = EG \\ \text{and } \angle BEG = \angle FEG \end{array} \right\} \text{................Const.}$$

$\therefore \quad BG = GF$........................I. 4.

Similarly, $BH = HF$.

But AH, HF are g^r than AF................I. 20.

i.e. AH, HB are g^r than AG, GB.

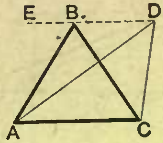
Wherefore, *G is the point required.*

Q.E.F.

V. Theorem.

The perimeter of a triangle of given area, and on a given base, is a minimum when the triangle is isosceles.

Let ABC be an isosceles $\triangle$, and ADC another equal $\triangle$, on the same base AC;



The perimeter of ABC shall be less than the perimeter of ADC.

Construction—

Join BD, and produce it to E.

Proof—

$\because$ $\triangle ABC = \triangle ADC$ ....Given.
$\therefore$ ED is $\|^{l}$ to AC ....I. 39.

$\because$ $AB = BC$, ...Given.
$\therefore$ $\angle BAC = \angle BCA$ ....I. 5.
$\therefore$ $\angle EBA = \angle DBC$ ....Ax. 1, I. 29.
$\therefore$ the sum of AB, BC is a minimum......App. iv.

Wherefore, *the perimeter, etc.*

Q.E.D.

Note.

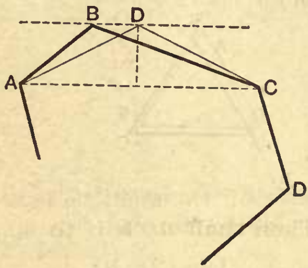
From this we infer, that of all triangles of equal area, the equilateral triangle has the minimum perimeter.

And, conversely, of all triangles of given perimeter, the equilateral triangle has the greatest area.

VI. Theorem.

If a polygon be not equilateral, an equal polygon with the same number of sides may be formed, having a less perimeter.

Let $ABCD$ ... be a polygon which is not equilateral, and AB, BC be two unequal adjacent sides;



It is required to show that a polygon of equal area to $ABCD$...... can be formed, with a less perimeter.

Construction—

Join AC.

On AC make an isosceles $\triangle$, equiareal to ABC...I. 31, 10, 11

Proof—

$\because$ AD, DC are less than AB, BC..........App. v.

$\therefore$ if ADC be substituted for ABC, the new polygon will be unchanged in area, but will have a less perimeter.

Wherefore, *if a polygon, etc.*

Q.E.D.

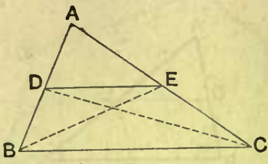
Note.

This Proposition leads us to the conclusion that "the perimeter of a polygon of given area, and given number of sides, is a minimum when the polygon is equilateral."

VII. Theorem.

The straight line joining the mid points of two sides of a triangle is parallel to the base; and conversely, if a straight line be drawn through the mid point of one side of a triangle parallel to the base, it bisects the other side.

1. Let ABC be a $\triangle$, and DE a st. line joining the mid points of the sides;



Then shall DE be $\|^l$ to BC.

Construction— Join CD, BE.

Proof— $\because\ AD = DB$ .... Given.
$\therefore\ \triangle BDE = \triangle ADE$ .... I. 38.
Similarly $\triangle CED = \triangle ADE$,
$\therefore\ \triangle BDE = \triangle CDE$ .... Ax. 1.
And they are on the same base DE,
$\therefore\ BC$ is $\|^l$ to DE .... I. 39.

2. Let D be the mid pt. of AB, and DE $\|^l$ to BC.

Then shall $AE = EC$.

Proof— $\because\ AD = DB$ .... Given.
$\therefore\ \triangle BDE = \triangle ADE$ .... I. 38.
$\because\ DE$ is $\|^l$ to BC .... Given.
$\therefore\ \triangle BDE = \triangle CED$ .... I. 37.
$\therefore\ \triangle ADE = \triangle CED$ .... Ax. 1.
Whence we can easily show by a "Reductio ad absurdum" that $AE = CE$.

Wherefore, *the straight line joining, etc.*

Q.E.D.

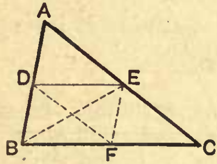
NOTE.

The converse may also be proved by means of Playfair's Axiom.

VIII. THEOREM.

The straight line joining the mid points of the sides of a triangle is equal to half the base, and cuts off a quarter of the triangle.

Let ABC be a $\triangle$, and D, E, mid pts. of AB, AC;



Then shall (1) $DE =$ **half** BC,
(2) $ADE = \frac{1}{4}$ **triangle** ABC.

CONSTRUCTION—

Draw $EF \parallel^l$ to AB, meeting BC in F .... I. 31.
Join DF, BE.

PROOF—

1. $DBFE$ is a $\square^{gram}$ .... App. vii.
$\therefore \quad EF = DB$ .... I. 34.
$= AD$ .... Const.
$\therefore \quad AE$ is $\parallel^l$ to DF .... I. 33.
$\therefore \quad DECF$ is a $\square^{gram}$.
Hence $\left.\begin{aligned} BF &= DE \\ &= FC \end{aligned}\right\}$ .... I. 34.
$\therefore \quad DE =$ half BC.

2. $\triangle ADE = \triangle DBE$ .... I. 38.
$= \frac{1}{2} \triangle ABE$,
$\triangle ABE = \triangle EBC$ .... I. 38.
$\therefore \quad \triangle ADE = \frac{1}{4} ABC$.

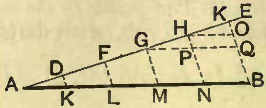
WHEREFORE, *the straight line, etc.*

Q.E.D.

IX. Problem.

To divide a straight line into any number of equal parts.

Let AB be the given st. line;



It is required to divide it into a given number of equal parts (say 5).

Construction—

From A draw a st. line AE of unlimited length, making a small acute $\angle$ with AB.

In AE take a pt. D, and mark off DF, FG, GH, HK, all equal to AD........................I. 3.

Join BK.

Through D, F, G, H draw st. lines $\|^l$ to BK, meeting AB in K, L, M, N respectively........................I. 31.

Through G draw GPQ $\|^l$ to AB........................I. 31.

AB shall be divided equally at K, L, M, N.

Proof—

$\because$ $GH = HK$, and HP is $\|^l$ to KQ, ...Const.

$\therefore$ $GP = PQ$........................App. vii.

But GN, PB are $\square^{\text{grams}}$........................Const.

$\therefore$ $MN = NB$........................I. 34.

In the same way we can show that all the other parts are equal.

Wherefore, *the straight line has been divided as required.*

Q.E.F.

Notes.

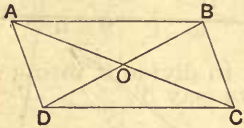
For another method, when the required number of parts is 3, see Appendix xvii.

If the number of parts required is a power of 2, Euc. I. 10 may be applied.

X. Theorem.

The diagonals of a parallelogram bisect one another; and conversely, if the diagonals of a quadrilateral bisect one another, it is a parallelogram.

1. Let $ABCD$ be a ▱$^{\text{gram}}$, with diag$^{\text{ls}}$ AOC, BOD;



Then shall $AO = OC$ and $BO = OD$.

Proof—

$\because$ in the $\triangle^{s}$ AOB, COD

$$\text{we have}\begin{cases} \left.\begin{array}{l}\angle OBA = \angle ODC \\ \angle OAB = \angle OCD\end{array}\right\} \text{.................... I. 29.} \\ \text{and } AB = DC \text{........................ I. 34.}\end{cases}$$

$$\left.\begin{array}{r}\therefore\ AO = OC \\ \text{and } BO = OD\end{array}\right\} \text{...................... I. 26.}$$

Which proves Part I.

2. Let $AO = OC$ and $BO = OD$;

Then shall the figure $ABCD$ be a ▱gram.

Proof—

$\because$ in the $\triangle^{s}$ AOB, COD

$$\text{we have}\begin{cases} \left.\begin{array}{r}AO = OC \\ BO = OD\end{array}\right\} \text{.................... Given.} \\ \text{and } \angle AOB = \angle COD \text{.................... I. 15.}\end{cases}$$

$\therefore\ \angle ABO = \angle ODC$...................... I. 4.

$\therefore\ AB$ is $\|^{l}$ to CD...................... I. 27.

Similarly AD is $\|^{l}$ to BC,

$\therefore\ ABCD$ is a parallelogram.

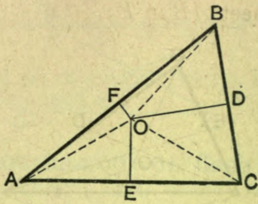
Wherefore, *the diagonals, etc.*

Q.E.D.

XI. Theorem.

The straight lines drawn perpendicular to the sides of a triangle from their mid points, are concurrent (i.e., meet in a point).

Let ABC be a $\triangle$, D, E, F, the mid points of the sides, and let the $\perp^{rs}$ EO, DO meet in O, and FO be joined;



Then shall FO be $\perp^r$ to AB.

Construction—

Join OA, OB, OC.

Proof— 1. In the $\triangle^s$ AEO, CEO,

$$\because \left\{ \begin{array}{c} AE = CE \\ EO = EO \\ \angle AEO = \angle CEO \end{array} \right\} \text{.................... Given.}$$

$\therefore \quad AO = OC$ .... I. 4.

Similarly $BO = OC$.

$\therefore \quad AO = BO$ .... Ax. 1.

2. In the $\triangle^s$ AOF, BOF,

$$\because \left\{ \begin{array}{l} AO = BO \\ OF = OF \\ FA = FB \text{ .... Given.} \end{array} \right.$$

$\therefore \quad \angle OFA = \angle OFB$ .... I. 8.

$\therefore \quad OF$ is $\perp^r$ to AB .... Def. 10.

Wherefore, *the straight lines, etc.*

Q.E.D.

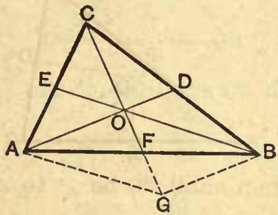
Note.

This Construction enables us to find a point equidistant from three given points.

XII. Theorem.

The medians of a triangle are concurrent.
(*I.e., the straight lines drawn from the angular points, to the mid points of the opposite sides*).

Let ABC be a $\triangle$, and D, E the mid points of the sides BC, CA. Let AD, BE meet in O, and CO be joined, and produced to meet AB in F;



Then shall $AF = FB$.

Construction—

Through A draw AG $\|^l$ to BE,..............I. 31.
and let it meet CO produced, in G.
Join BG.

Proof—

$\because\ AE = EC$........................Given.
and EO is $\|^l$ to AG......................Const.
$\therefore\ CO = OG$ ....App. vii

$\because\ CO = OG$,
and $CD = DB$........................Given.
$\therefore\ OD$ is $\|^l$ to GB.................App. vii.

$\therefore\ AOBG$ is a $\square^{\text{gram}}$.
$\therefore\ AF = FB$.......................App. x.

Wherefore, *the medians of a triangle are concurrent.*

Q.E.D.

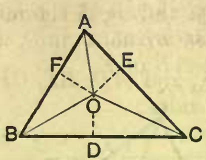
Corollary—

$\because\ OF = FG$ ....App. x.
and $CO = OG$ ....Proved above.
$\therefore\ OF = \frac{1}{3}\ CF$.
So for the other medians.

XIII. THEOREM.

The angle-bisectors of a triangle are concurrent.

Let ABC be a $\triangle$, and let the angle-bisectors AO, BO meet in O;



Then CO shall bisect the $\angle ACB$.

CONSTRUCTION—

From O, draw OD, OE, OF, $\perp^r$ to the sides.......I. 12.

PROOF—

1. In the $\triangle^s$ BOF, DOF,

$$\because \begin{cases} \angle FBO = \angle DBO \text{.....................Given.} \\ \angle OFB = \angle ODB \text{.....................Const.} \\ OB = OB. \end{cases}$$

$\therefore \quad OD = OF$...........................I. 26.

Similarly $OF = OE$,

$\therefore \quad OD = OE$.........................Ax. 1.

2. Sqs. on OE, EC = sq. on OC

= sqs. on OD, DC...........I. 47.

But sq. on OE = sq. on OD,

$\therefore$ sq. on CE = sq. on CD.................Ax. 3.

$\therefore \quad CE = CD$.

3. In the $\triangle^s$ OEC, ODC,

$$\because \begin{cases} OE = OD, \\ EC = CD, \\ OC = OC, \end{cases}$$

$\therefore \quad \angle OCD = \angle OCE$..........................I. 8.

WHEREFORE, *the angle-bisectors of a triangle are concurrent.*

Q.E.D.

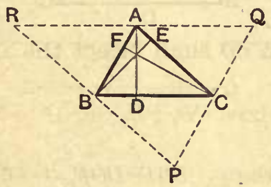
NOTE.

This construction enables us to find a point equidistant from three given lines.

XIV. THEOREM.

The perpendiculars from the angular points of a triangle on the opposite sides are concurrent.

Let ABC be a $\triangle$, and AD, BE, CF $\perp^{rs}$ from A, B, C on the opposite sides;



Then shall AD, BE, CF, be concurrent.

CONSTRUCTION—

Through A, B, C draw $\|^{ls}$ to the opposite sides, meeting each other in P, Q, R.....I. 31.

PROOF—

$\because$ $ABPC$ and $ABCQ$ are $\square^{grams}$,
$\therefore$ $PC = AB = CQ$ ....I. 34.
i.e., PQ is bisected in C.

Similarly A, and B, are the mid pts. of QR, PR,
$\therefore$ the $\perp^{rs}$ at these pts. are concurrent.......App. xi.
and AD, BE, CF, are these $\perp^{rs}$..............I. 29.

WHEREFORE, *the perpendiculars, etc.*

Q.E.D.

NOTE.

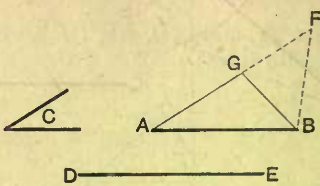
The point in which these perpendiculars meet, is called the *Orthocentre* of the triangle.

The point in which the "medians" intersect, is called the Centre of Gravity of the triangle.

XV. Problem.

To construct a triangle, having given the base, one of the angles at the base, and the sum of the sides.

Let AB be the base, C the given $\angle$, and DE the sum of the sides.



Construction—

At the pt. A make the $\angle BAF = \angle C$............I. 23.

Make $AF = DE$..........................I. 3.

Join BF.

At the pt. B make the $\angle FBG = \angle GFB$..........I. 23.

Let BG meet AF in G.

ABG **shall be the $\triangle$ required.**

Proof—

$\because \angle GBF = \angle GFB$ ....Const.

$\therefore GB = GF$...........................I. 6.

$\therefore AG, GB = AF = DE$ ....Ax. 2.

and $\angle GAB = \angle C$..........................Const.

Wherefore, *a triangle has been constructed, etc.*

Q.E.F.

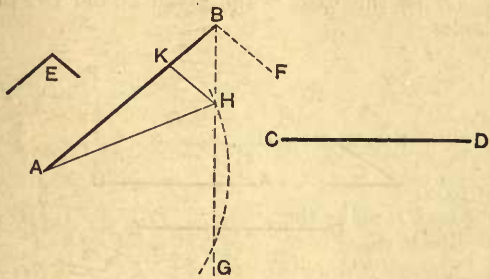
Note.

The construction for any given problem may often be best found by Analysis. In the above case let us assume the triangle AGB made as required. On producing AG to F so that $AF = DE$, and joining BF, we see that $GF = GB$. Hence we see that the $\angle GFB = \angle GBF$. We can draw the $\angle GFB$, hence we can construct the $\angle FBG$, and so we deduce the above Construction.

XVI. Problem.

To construct a triangle, having given the base, vertical angle, and sum of the sides.

Let AB be sum of sides, CD base, E vertical $\angle$.



Construction—At B make $\angle ABF = \angle E$..................... I. 23.

Bisect the $\angle ABF$ by BG..................... I. 9.

From A with rad. CD describe an arc cutting BG in H.

From H draw $HK \parallel^{l}$ to BF, meeting AB in K...I. 31.

Join AH.

AHK **shall be the $\triangle$ required.**

Proof— $\angle KBH = \angle HBF$..................Const.

$= \angle KHB$..................I. 29.

$\therefore\ KB = KH$......................I. 6.

$\therefore\ AK,\ KH = AB$,

and $\angle AKH = \angle KBF = \angle E$..........I. 29. Const.

and $AH = CD$.

Wherefore, *a triangle has been constructed as required.*

Q.E.F.

Analysis.

Suppose AKH the triangle drawn as required.
Produce AK to B so that $KB = KH$,
Then $\angle KHB = \angle KBH = \frac{1}{2}$ ext^{r} $\angle AKH$....................I, 5, 16.
Hence we must make an $\angle = E$, at B and bisect it, and so on, as above.

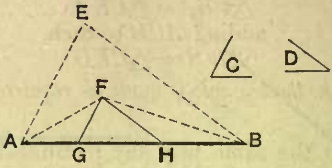
Note.

If one of the sides (as AK) were given, instead of the sum, we should make the angle AKH equal to the given angle E, and describe the arc to cut KH.

XVII. Problem.

To construct a triangle, having given the perimeter, and two angles.

Let AB be the perimeter, C and D the $\angle^s$.



Construction—

At A and B make the $\angle^s$ BAE, $ABE = \angle^s$ C and D...I. 23.

Bisect these $\angle^s$ by AF, BF, meeting in F.........I. 9.

Through F draw FG, FH $\|^l$ to AE, and BE......I. 31.

FGH **shall be the $\triangle$ required.**

Proof— $\because$ $\angle GAF = \angle EAF = \angle AFG$..............Const. I. 29.

$\therefore$ $GA = GF$..........................I. 6.

Similarly $HB = HF$,

$\therefore$ the perimeter of $FGH = AB$.

also $\angle FGH = \angle EAG = \angle C$
and $\angle FHG = \angle EBH = \angle D$ } ....I. 29. Const.

Wherefore, *the triangle is constructed as required.*

Q.E.F.

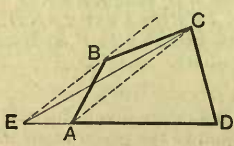
Note.

If the triangle be equilateral, the above gives us a method of trisecting the given straight line AB.

XVIII. Problem.

To make a triangle equal to a given quadrilateral.

Let $ABCD$ be the given quadrilateral.



CONSTRUCTION—

Join AC.

Through B draw $BE \parallel^l$ to AC, meeting DA produced in E, and join EC..................I. 31.

CDE shall be the $\triangle$ required.

PROOF— $\triangle CBA = \triangle CEA$....................I. 37.

$\therefore$ adding ACD to each,

$ABCD = \triangle CED$.

WHEREFORE, *the triangle is made as required.*

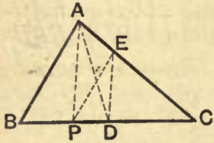
Q.E.F.

COROLLARY.—In the same way any rectilineal figure may be reduced to a triangle, by reducing the number of sides one at a time. *E.g.*, a pentagon may be reduced to a quadrilateral, and this to a triangle, by two applications of the above construction.

XIX. PROBLEM.

To bisect a triangle by a straight line drawn through a given point in one of its sides.

Let ABC be the given $\triangle$, and P the given pt. in BC.



CONSTRUCTION— Join AP.

Bisect BC in D ....I. 10.

Through D draw $DE \parallel^l$ to AP, meeting AC in E...I. 31.

Join PE.

PE shall be the line required.

PROOF— $\triangle ADP = \triangle APE$....................I. 37.

$\therefore$ adding APB to each,

$ABPE = \triangle ABD$,

$= \frac{1}{2} \triangle ABC$..................I. 38.

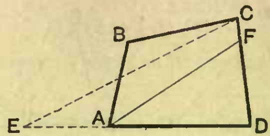
WHEREFORE, *the triangle ABC is bisected by PE.*

Q.E.F.

XX. Problem.

To bisect a quadrilateral by a straight line drawn through an angular point.

Let $ABCD$ be the quadrilateral, A the given point;



It is required to bisect $ABCD$ by a straight line drawn through A.

Construction—

Make a $\triangle CED$ equal to $ABCD$, having its vertex at C, and its base AD produced ....App. xviii.

Bisect this $\triangle$ by a st. line AF through A...App. xix.

AF shall be the st. line required.

Proof—

$AFD = \frac{1}{2}\, CED$ ....Const.

$= \frac{1}{2}\, ABCD$....................Const.

Wherefore, *the quadrilateral is bisected as required.*

Q.E.F.

Note.

If, with the above Construction, AF lies on the other side of AC, and so does not cut CD, the auxiliary triangle must be made with its vertex at C, and its base in BA produced.

XXI. Theorem.

If two triangles have one angle of the one equal to one angle of the other, and the second angles in both triangles either both acute, or right, or obtuse, and the sides containing the third angles equal, each to each; then the two triangles shall be equal in all respects.

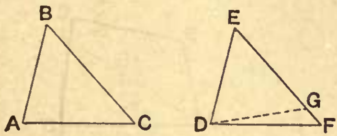
Let the $\triangle^s$ ABC, DEF have

$BA = ED$,

$AC = DF$,

and $\angle ABC = \angle DEF$,

and the $\angle^s$ ACB, DFE, either both acute, both right, or both obtuse ;



Then they shall be equal in all respects.

CONSTRUCTION—

Let the $\triangle$ BAC be placed on the $\triangle EDF$ so that BA coincides with ED..............($\because BA = ED$).

PROOF—

$\because$ $\angle ABC = \angle DEF$....................Given.

$\therefore$ BC will lie on EF,

Now, if C do not coincide with F, suppose the $\triangle$ to take the position EDG.

Then $DG = DF$.......................Given.

$\therefore$ $\angle DGF = \angle DFG$......................I. 5.

1st. Suppose BCA and DFE both obtuse, or right $\angle^s$.

Then the above is impossible..................I. 17.

2nd. Suppose BCA and DFE both acute $\angle^s$.

$\therefore$ DGF is an acute $\angle$.

$\therefore$ DGE is an obtuse $\angle$..................I. 13.

But it is also an acute $\angle$.................($\because$ it $= ACB$).

Which is impossible.

$\therefore$ BC must coincide with EF and AC with DF.

$\therefore$ the $\triangle^s$ are equal in all respects.

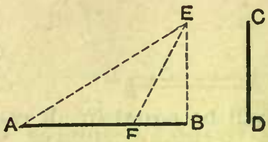
WHEREFORE, *if two triangles, etc.*

Q.E.D.

XXII. PROBLEM.

To divide a given straight line into two parts, the difference of whose squares shall be equal to the square on a given line.

Let AB be the line to be divided, CD the other line;



It is required to divide AB (at F, say), so that the difference of the squares on AF, FB may be equal to the square on CD.

CONSTRUCTION—

From B draw BE $\perp^r$ to AB..................App. i.
Make BE equal to CD....................I. 3.
Join AE.
At E make $\angle AEF = \angle BAE$, and let EF meet AB in F...I. 23.

AB shall be divided at F as required.

PROOF—

$\because$ $\angle EAF = \angle AEF$....................Const.
$\therefore$ $AF = FE$..........................I. 6.
$\therefore$ sq. on AF = sq. on FE.
= sqs. on EB, BF.........I. 47.
$\therefore$ difference of sqs. on AF, FB = sq. on EB
= sq. on CD.........Const.

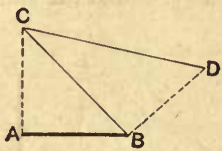
WHEREFORE, *AB is divided as required.*

Q.E.F.

XXIII. Problem.

To make a square which shall be double, treble, etc., of a given square.

Let AB be a side of the given square;



It is required to make squares which shall be respectively double, and treble, of the square on AB.

Construction—

At A draw $AC \perp^r$ to AB and equal to AB......App. i.
Join BC.
At B draw $BD \perp^r$ to BC and equal to AB........App. i.
Join CD.

Then the square on BC shall be double, and the square on CD treble, the square on AB.

Proof—

Sq. on BC = sqs. on BA, AC.................I. 47.
= twice sq. on BA...............Const.

sq. on DC = sqs. on BC, BD.................I. 47
= thrice sq. on BA............. Const.

In the same way we could proceed to make a square any number of times the area of a given square.

Q.E.F.

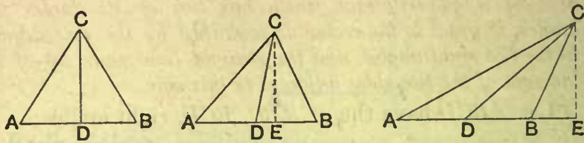
Note.

To make a square 4 times, 9 times, 16 times, etc., a given square, draw a straight line 2, 3, 4, etc., times as long. It will be the side of the square required. See II. 4., Cor. II.

XXIV. THEOREM.

The sum of the squares on the sides of a triangle is equal to twice the square on half the base, together with twice the square on the straight line joining the vertex to the mid point of the base.

Let ABC be a $\triangle$, and D the mid pt. of AB;



Then shall sqs. on AC, CB = twice sqs. on CD, DB.

CONSTRUCTION—

From C draw CE $\perp^r$ to AB, or AB produced .....I. 12.

PROOF—

First, let CE coincide with CD.

Then sqs. on AC, CB = $\left\{\begin{array}{l}\text{sqs. on } CD, DA, \\ \text{with sqs. on } CD, DB\end{array}\right.$ ....I. 47.

= twice sqs. on CD, DB ($\because DA = DB$).

Secondly, let CE not coincide with CD.

Then one of the $\angle^s$ ADC, BDC must be obtuse.....I. 13.

Let it be ADC.

Sq. on AC = $\left\{\begin{array}{l}\text{sqs. on } AD, DC, \\ \text{with twice rect. } AD.DE\end{array}\right.$ ....II. 12.

$\left.\begin{array}{l}\text{Sq. on } BC, \\ \text{with twice rect. } BD.DE\end{array}\right\}$ = sqs. on BD, DC........II. 13.

$\therefore$ adding equal to equals,

$\left.\begin{array}{l}\text{sqs. on } AC, CB \\ \text{with twice rect. } BD.DE\end{array}\right\} = \left\{\begin{array}{l}\text{twice sq. on } CD \text{, with sqs. on } AD, \\ DB \text{, and twice rect. } AD.DE.\end{array}\right.$

But rect. $BD.DE$ = rect. $AD.DE$ ....($\because AD = BD$).

$\therefore$ sqs. on AC, CB = twice sqs. on CD, DB......Ax. 3.

WHEREFORE, *the sum of the squares, etc.*

Q.E.D.

ALGEBRAIC SOLUTION.

Let $BC = a$, $AC = b$, $AB = 2c$, $CD = d$, $DE = x$.

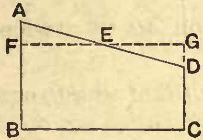
$$b^2 = c^2 + d^2 + 2cx,$$
$$a^2 = c^2 + d^2 - 2cx,$$
$$\therefore \quad a^2 + b^2 = 2(c^2 + d^2).$$

Q.E.D.

XXV.

The area of a quadrilateral which has two of its angles right angles, is equal to the rectangle contained by the side adjacent to the two right angles, and the straight line made up of half the sum of the two sides adjacent to this side.

Let $ABCD$ have the $\angle^s$ ABC, BCD, right angles;



Then the area of *ABCD* shall = rect. contained by *BC* and half the sum of *AB*, *CD*.

CONSTRUCTION—

Bisect AD in E .... I. 10.

Through E draw FEG $\|^l$ to BC .... I. 31.

Meeting AB, and CD produced, in F and G.

PROOF— In the $\triangle^s$ EAF, EDG.

$$\because \begin{cases} \angle EAF = \angle EDG \\ \angle EFA = \angle EGD \end{cases} \text{ .... I. 29.}$$
and $AE = ED$ .... Const.

$$\therefore \quad \left.\begin{array}{l} \triangle AFE = \triangle EGD, \\ \text{and } AF = GD. \end{array}\right\} \text{ .... I. 26.}$$

$\therefore$ area of $ABCD$ = area of rect. $FBCG$
= rect. $BC.CG$.

And $\because$ $AF = GD$,
and $FB = CG$ .... I. 34.

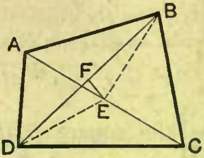
$\therefore$ CG = half the sum of AB, CD.

WHEREFORE, *the area of a quadrilateral, etc.* Q.E.D.

XXVI. Theorem.

The squares on the sides of a quadrilateral are together equal to the sum of the squares on its diagonals, together with four times the square on the straight line joining the mid points of the diagonals.

Let $ABCD$ be a quadrilateral, and E, F the mid pts. of AC, BD;



Sqs. on AB, BC, CD, DA shall = sqs. on AC, BD, with four times sq. on EF.

Construction—

Join DE, BE.

Proof—

$\because$ Sqs. on AB, BC = twice sqs. on BE, EC }
Sqs. on AD, DC = twice sqs. on DE, EC } ..App. xxiv.

$\therefore$ Sqs. on AB, BC, CD, DA = { twice sqs. on BE, ED, with 4 times sq. on EC......Ax. 2.

= { 4 times sqs. on EF, FB.App. xxiv. with sq. on AC......II. 4. Cor. II.

= { 4 times sq. on EF, with sqs. on BD and AC...II. 4. Cor. II.

Wherefore, *the squares on the sides, etc.*

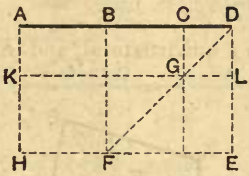
Q.E.D.

Corollary—

Hence, the squares on the sides of a parallelogram are together equal to the squares on the diagonals (App. x.)

XXVII. Theorem.

If a straight line AD be divided at any two points B and C, then rects. $AB \cdot CD$ and $BC \cdot AD$ = rect. $AC \cdot BD$.



Construction—

On BD describe the square $BDEF$, and complete the figure as in II. 5.

Proof —

Rects. $AB \cdot CD$ and $BC \cdot AD$ = rects. $AB \cdot KA$ and $KL \cdot EL$
= KB and KE
= KB, HG and GE
= KB, HG and BG...........I. 43.
= HC.
= rect. $AC \cdot BD$.

Wherefore, *if a straight line, etc.*

Q.E.D.

Algebraic Proof.

Let $AB = a$, $BC = b$, $CD = c$,
Then we have to prove that
$ac + b\ (a + b + c) = (a + b)(b + c)$,
Which is at once evident on multiplication.

Q.E.D.

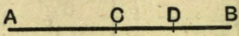
Note.

This Theorem is true for any position of the points B, C, whether in AD, or AD produced; regard being had to sign. See Notes on II. 5-8.

XXVIII. Problem.

To divide a straight line into two parts so that the rectangle contained by them shall be a maximum.

Let AB be the given st. line.



Construction—

Bisect AB at C........................I. 10.
and in it take any other pt. D.

The rectangle $AC \cdot CB$ shall be a maximum.

Proof—

Rect. $AD \cdot DB$ with sq. on CD = sq. on CB.......II. 5.
= rect. $AC \cdot CB$.

$\therefore$ Rect. $AD \cdot DB$ is less than rect. $AC \cdot CB$.

Wherefore, *AB is divided at C as required.*

Q.E.F.

Corollary—

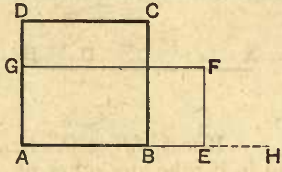
$$\left.\begin{array}{r}\because \text{ sqs. on the two parts of } AB \\ \text{with twice rect. contained by the parts}\end{array}\right\} = \text{sq. on } AB,$$

It is evident that AB is divided at C, so that the sum of the sqs. on the two parts is a minimum.

XXIX. THEOREM.

Of all rectangles of given area, the square has the least perimeter.

Let $ABCD$, $AEFG$ be a sq. and rect. of equal area;



The perimeter of the sq. shall be less than that of the rect.

CONSTRUCTION—

Produce ABE to H, so that $BH = AB$.

PROOF—

$\because \quad AB = BH.$

sq. on AB is greater than rect. $AE \cdot EH$...App. xxviii.

But sq. on AB = rect. $AE \cdot EF$.

$\therefore \quad EF$ is greater than EH.

$\therefore \quad AE, EF$ are tog^r greater than twice AB.

$\therefore \quad$ perim^r of AF is greater than perim^r of AC.

WHEREFORE, *of all rectangles, etc.*

Q.E.D.

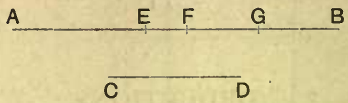
NOTE.

This Proposition is the converse of the preceding one.

XXX. Problem.

To divide a straight line into two parts, so that the rectangle contained by them may be equal to a given square.

Let AB be the st. line, CD a side of the given sq.



Construction—

From AB cut off AE equal to CD............I. 3.

Bisect AB in F........................I. 10.

Take FG such that AE, FG may be the sides of a right-angled $\triangle$ whose hypoteneuse is AF....................................App. xvi. Note.

Then AB shall be divided at G as required.

Proof—

Sq. on AF = sqs. on AE, FG..............Const.

But sq. on AF = rect. $AG \cdot GB$ with sq. on FG....II. 5.

$\therefore$ rect. $AG \cdot GB$ with sq. on FG = sqs. on AE, FG...Ax. 1.

$\therefore$ rect. $AG \cdot GB$ = sq. on AE.........Ax. 3.

= sq. on CD.

Wherefore, *AB is divided at G as required.*

Q.E.F.

Note.

From Proposition xxviii., we see that CD must be less than half the given line AB.

DEDUCTIONS.

THESE Exercises are divided into three classes, and arranged, as far as possible, in order of difficulty. In the first class, references are given to all the Propositions, in order, which are needed for the construction and proof of each Deduction. In the second class, only the leading Proposition on which the Deduction depends, is given, as is usually done in Examination Papers; while, in the third class, the student is left to himself. These references are all so printed that they may be removed if it is thought advisable.

Deductions which are analogous to one another are not placed together, as a rule, but with some five or six exercises between them, in order that the learner may be occasionally reminded of his back work. When a reference is given to a former Deduction, it is to show that a similar method is to be again employed, not that the former result should be quoted merely.

BOOK I.

A.

I. 3. 1. On a given straight line as base, describe an isosceles
Post. 3, 1. triangle, having each of the sides double of the base.

I. 4. 2. If two straight lines bisect each other at right angles, and any point in one of them be joined to the ends of the other, these joining lines are equal.

I. 4. 3. The sides AB, AD of a quadrilateral are equal, and the diagonal AC bisects the angle BAD; show that the sides CB and CD are equal, and that the diagonal AC bisects the angle BCD.

I. 15, 4. 4. Two triangles BCD, FCD stand on the same side of the base CD, and the sides BD, FC mutually bisect each other. Prove that BF is equal to CD.

I. 5. Ax. 2. 5. The opposite angles of a rhombus are equal.

I. 5. 6. In the figure of I. 5, prove that the angle VMQ is equal to the angle QRV.

I. 1, 5. 7. In the figure of I. 1, if the circles cut at O and P, prove that the angle PKO is equal to the angle PMO.

I. 5. 8. Two circles whose centres are O and Q, cut at P and R; show that the angle OPQ is equal to the angle ORQ.

I. 8. 9. ABC is an isosceles triangle; D is the middle point of the base BC; prove that AD bisects the vertical angle BAC.

I. 8. 10. A point A is taken in the circumference of a circle whose centre is O, and a circle is described having A as centre, and meeting the first circle in B and C; prove that AO bisects the angle BAC.

I. 8. 11. A diagonal of a rhombus bisects each of the angles through which it passes.

I. 16, 4. 12. If in the figure of Proposition XVI., DK be joined, then DK is equal to EF.

I. 17, 13. 13. How many of the exterior angles of any triangle must be obtuse?

I. 21. 14. Two triangles on the same base are such that one lies wholly inside the other; prove that the inner one has the smaller perimeter.

I. 8, 4. 15. The diagonals of a rhombus are at right angles.

16. If the lengths of two sides of a triangle be 3 inches and 4 inches respectively, between what limits must the length of the third side lie? I. 20.

17. If two isosceles triangles are on the same base, the straight line joining their vertices is perpendicular to the base. I. 8, 4.

18. In the figure of I. 5, if RQ and VM meet in O, prove that OR is equal to OM. I. 5, 6.

19. If BAC, BAD be two triangles on the same base AB, with the angle BAC equal to the angle ABD, and ABC equal to BAD, then the triangles BDC, ADC are equal in all respects. I. 26.

20. Any point on the line bisecting an angle, is equidistant from the arms of the angle. I. 12, 26.

21. Straight lines bisecting two adjacent angles of a parallelogram intersect at right angles. I. 29, 32.

22. What is the size of the angle of a regular octagon. I.32,Cor.1

23. If two equal triangles are between the same parallels, they are on the same base, or on equal bases. I. 3, 37.

24. Straight lines which are at right angles to the same straight line are parallel to each other. I. 28.

25. AB, BC, CD are three equal straight lines. The angle ABC is greater than BCD. Prove that AC is greater than BD. I. 24.

26. If the equal sides AB, AC of an isosceles triangle be produced, as in I. 5, to F and G, so that AF is equal to AG, and if BG, CF intersect in H, then AH will bisect the angle BAC. I. 5, 6, 8.

27. The angle A of a triangle is bisected by AD meeting BC at D, prove that AB is greater than BD. I. 16, 19.

28. If the sides AB, BC, CD, DA of a quadrilateral are in descending order of magnitude, then the angle CDA is greater than CBA. I. 18.

29. Prove that the common chord of two equal circles which cut one another, bisects the line which joins their centres, at right angles. I. 8, 4.

30. If the angles on the other side the base of a triangle, whose sides have been produced, are equal, prove that the triangle is isosceles. I. 13, 6.

31. If an isosceles triangle has each base angle double of the vertical, find their size. I. 32.

32. Prove that the shortest line that can be drawn with its ends on the circumferences of two concentric circles, is one which, when produced, passes through the centre. I. 20.

33. If one diagonal of a quadrilateral bisect the two angles at its ends, it will bisect the other diagonal at right angles. I. 26, 4.

34. The diagonals of a parallelogram bisect each other. I. 29, 34, 26

35. If one straight line stand upon another straight line, the bisectors of the adjacent angles are perpendicular to each other. I. 13.

I. 29, 34, 26, 38. 36. The diagonals of a parallelogram divide it into four equal parts.

I. 4, 15, 28. 37. If the diagonals of a quadrilateral bisect each other, it is a parallelogram.

I. 3, 31. Ded. 34. 38. AB, AC are two given straight lines: through a given point E between them, it is required to draw a straight line GEH, such that the intercepted portion GH shall be bisected at the point E.

I. 26, 34. 39. If a straight line joining two opposite angles of a parallelogram bisect the angles, the quadrilateral is a square or a rhombus.

I. 9, 31. Ded. 39, 15 40. Construct a rhombus, having given the length of a diagonal, and one of the angles through which it passes.

I. 23, 32. 41. On a given straight line construct a triangle equiangular to a given triangle.

I. 13, 32. 42. D is a point on the side BC of a triangle ABC. If the angles ADC, ADB are respectively double of the angles ABC, ACB, the triangle is right-angled.

I. 3, 10, 38. 43. Given a triangle, describe another, such that four times the latter is equal to five times the former.

I. 34. 44. If two straight lines BA, BC, be respectively parallel to two others DE, DF, the angle CBA is equal to the angle EDF.

I. 1, 31, 32. 45. On a given straight line describe a rhombus having one angle equal to two-thirds of a right angle.

I. 20. 46. The four sides of any quadrilateral are greater than the two diagonals together.

I. 29, 5. 47. Any straight line parallel to the base of an isosceles triangle makes equal angles with the sides.

I. 34, 8. 48. If the diagonals of a parallelogram are equal, all its angles are equal.

I. 32, Cor. 1 What is the size of each angle?

I. 31, 34, 35 49. Construct a rhombus equal to a given parallelogram.

Ded. 43. 50. Make a triangle five-eighths the area of a given triangle.

I. 8, 6. 51. ACB, ADB are two triangles on the same base AB, and on the same side of it; AC is equal to BD, and AD to BC, and AD, BC meet in O; prove that the triangles OAB, OCD are isosceles.

I. 16. 52. In the figure of I. 5, if RQ and VM meet in H, prove that the angle VHQ is greater than RKM.

I. 16, 5, 19. 53. The straight line joining the vertex of an isosceles triangle to any point in its base, is always less than one of the equal sides.

I. 23. 54. Given three sides of a quadrilateral, and the angles adjacent to one of them, construct it.

I. 12, 3, 4. 55. Construct an angle double of a given angle, without using any Propositions beyond the XIIth.

56. AB and CD are two diameters of a circle; prove that if C and D be joined to B, the straight lines CB, DB will bisect the angles made with AB by a straight line through B parallel to CD. I. 29, 5.

57. ABC is a triangle and P any point within it. Show that the sum of PA, PB, PC is less than the sum of the sides of the triangle. I. 21.

58. If the opposite sides of a quadrilateral be equal, it is a parallelogram. I. 8, 28.

59. If the opposite angles of a quadrilateral be equal, it is a parallelogram. I. 32, 28.

60. A is the vertex of an isosceles triangle ABC, and BA is produced to D so that AD is equal to AB, and DC is joined; show that BCD is a right angle. I. 5, 32.

61. $ABCD$ is a quadrilateral with BC parallel to AD; show that its area is the same as that of the parallelogram which can be formed by drawing through the middle point of CD, a straight line parallel to AB. I. 29, 26.

62. In the figure of I. 21, the difference between the angles BDC, BAC is equal to the sum of the angles ABD, ACD. I. 32.

63. ABC is a triangle. Through B, BD is drawn perpendicular to BC, and through A, AD perpendicular to BA; these lines meet in D. Prove that the angle ADB is equal to the angle ABC. I. 32.

64. Show how to make an angle equal to one-twelfth of a right angle. I. 1, 9, 32.

65. A square and a rhombus stand on the same base. Prove that the square is greater than the rhombus. I. 17, 19, 35

66. $ABCD$ is a quadrilateral with BC parallel to AD; E is the middle point of CD; show that the triangle AEB is half the quadrilateral. Ded. 61. I. 41.

67. AB, CD, EF are three equal and parallel straight lines; prove that the triangle ACE is equal to the triangle BDF in all respects. I. 33, 8.

68. Construct a triangle, having given the base, one of the angles at the base, and the sum of the sides. I. 23, 6.

69. If one of the equal sides of an isosceles triangle be produced beyond the vertex, and the exterior angle bisected, then the bisecting line is parallel to the base of the triangle. I. 32, 28.

70. Given three sides of a quadrilateral, and the angles adjacent to the fourth, construct it. I. 23, 3, 31, 34, 29.

B.

I. 5. 71. In the figure of I. 5, if RQ and VM meet in O, show that VO and QO are equal.

I. 19. 72. The perpendicular is the shortest straight line that can be drawn from a given point to a given straight line; and of others, that which is nearer to the perpendicular is less than the more remote; and two, and only two, equal straight lines can be drawn from the given point to the given straight line, one on each side of the perpendicular.

I. 47. 73. ABC is a triangle, and AD a perpendicular on BC; prove that the difference of the squares on AB, AC is equal to the difference of the squares on BD, DC.

I. 32. 74. If two straight lines be respectively perpendicular to two others, the angle between the former is equal to the angle between the latter.

I. 26. 75. A straight line is drawn terminated by two parallel straight lines; through its middle point any straight line is drawn and terminated by the parallel straight lines. Show that the second straight line is bisected at the middle point of the first.

I. 32. 76. From the ends of the base of an isosceles triangle straight lines are drawn perpendicular to the sides; show that the angles made by them with the base are each equal to half the vertical angle.

I. 32. 77. From a point P inside a triangle ABC, perpendiculars PM, PN are drawn to AB and AC; prove that MPN and MAN are together equal to two right angles.

I. 12. 78. Find a point equally distant from a given point and from a given straight line.

I. 10. 79. From the vertex of a scalene triangle draw a straight line to the base, which shall exceed the less side, by as much as it is exceeded by the greater.

I. 20. 80. The sum of the diagonals of a quadrilateral, is less than the sum of any four lines drawn to the four angles from any point within the figure, except their intersection.

Ded. 11. 81. The longer sides of a parallelogram are twice as long as the shorter sides. Show that the straight lines joining the middle point of one of the longer sides with the ends of the opposite side, are perpendicular to each other.

I. 31. 82. Find a point which is at a given distance from a given point, and from a given straight line.

I. 26. 83. $ABCD$ is a right-angled parallelogram, and AE, BF are drawn to meet the diagonals BD, AC in E and F respectively, so that the angle AEB is equal to the angle AFB. Prove that the triangles AEB, AFB are equal in all respects.

I. 5. 84. If AB and AC be equal sides of an isosceles triangle, and a circle with centre B, and radius BA, cut AC (or AC produced) in E; and BF be taken in AB (or AB produced) equal to CE; prove that the angle CFA is equal to the angle FAC.

85. Through each angular point of a triangle a straight line is drawn parallel to the opposite side. Show that the triangle thus obtained is equiangular with the given triangle. I. 29.

86. Any quadrilateral figure which is bisected by both its diagonals, is a parallelogram. I. 39.

87. If the straight line bisecting the exterior angle of a triangle be parallel to the base, the triangle is isosceles. I. 29.

88. On a given straight line as diagonal, describe a square. I. 32.

89. If two sides of a quadrilateral be parallel, and the middle points of the other two sides be joined, prove that this line is half the sum of the parallel sides. I. 31.

90. Find a point in a given straight line, such that its distances from two given points may be equal. I. 10.

91. If four straight lines meet at a point so that the opposite angles are equal, these straight lines are two and two in the same straight line. I. 15. Cor. 2.

92. In a given straight line find a point such that the perpendiculars drawn from it to two given straight lines shall be equal. Ded. 20.

93. If through any point equidistant from two parallel straight lines, two straight lines be drawn cutting the parallel straight lines, they will intercept equal portions of these parallel lines. I. 26.

94. Describe a circle which shall pass through two given points, and have its centre in a given line. I. 10.

95. Find a point B in a given straight line CD, such that if AB be drawn from a given point A, the angle ABC will be equal to a given angle. I. 31.

96. The bisectors of the base angles of an isosceles triangle contain an angle equal to an exterior angle of the triangle. I. 32.

97. Show that an angle of a triangle is obtuse, right, or acute, according as it is greater than, equal to, or less than the other two angles together. I. 32.

98. If A be the vertex of an isosceles triangle ABC, and CD be drawn perpendicular to AB, prove that the squares on the three sides are together equal to the square on BD, twice the square on AD, and thrice the square on CD. I. 47.

99. Through two points draw two straight lines, to form an equilateral triangle with a straight line given in position. Ded. 95.

100. Draw two squares, whose areas shall be in the ratio 5 to 7. App. 23.

101. Draw a straight line through a given point, such that the part of it intercepted between two given parallel straight lines, may be of given length. I. 34.

102. The figure formed by joining the middle points of the sides of any quadrilateral is a parallelogram, and its area half that of the quadrilateral. App. 7.

I. 26. 103. In the figure of I. 47, if DB, EC be produced to meet FG and HK (or either of these produced) in P and Q, show that BP and CQ are each equal to BC.

Ded. 72. 104. The diagonals of a rectangle meet in O. Prove that of all the straight lines drawn through O, and terminated by opposite sides, the diagonals are the greatest.

I. 8. 105. Two equal straight lines AB and CD are joined towards opposite parts by the equal straight lines AD and CB, intersecting in O. Prove that the triangles OAC, OBD are isosceles.

I. 32. 106. If ABC be a triangle, and through D, the middle point of AB, DE is drawn parallel to BC, and BE be drawn to bisect the angle ABC, and meet DE in E, AEB will be a right angle.

I. 34. 107. Draw a parallel to the base of a triangle, equal to a given straight line.

I. 47. 108. If perpendiculars be let fall on the sides of a triangle from any point within it, prove that the sums of the squares on alternate segments of the sides are equal.

App. 8. 109. If two triangles stand on the same base and on the same side of it, and the middle points of the sides be joined; the joining lines will form a parallelogram.

I. 37. 110. On the base of a given scalene triangle describe an isosceles triangle, equal to the given triangle.

App. 13. 111. The bisectors of two exterior angles, and of the third interior angle of a triangle, are concurrent.

I. 5. Cor. 112. The perimeter of the parallelogram formed by drawing parallels to two sides of an equilateral triangle from any point in the third side, is equal to twice the side.

Ded. 34. 113. $ABCD$ is a parallelogram. From a point P in BD, PA and PC are drawn; prove that the triangles PAB, PCB, are equal in area.

I. 20. 114. The difference of any two sides of a triangle is less than the third side.

I. 35. 115. If two parallelograms are on the same base, but the altitude of one is double that of the other, the area of the former is double that of the latter.

Ded. 15. 116. Through two given points in two parallels, draw two straight lines forming a rhombus with the parallels.

I. 34. 117. If two equal straight lines intersect each other anywhere at right angles, the quadrilateral formed by joining their ends is equal to half the square on either line.

I. 6. 118. A straight line is drawn, bisecting one of the angles of a rhomboid; prove that it forms with the sides of the rhomboid produced, three isosceles triangles.

I. 1. 119. Trisect a right angle.

I. 10. 120. Through a given point draw a straight line, such that the perpendiculars on it from two given points may be on opposite sides of it, and equal to each other.

121. In the triangle ABC, BC is bisected at E, and AB at G: AE is produced to F, so that EF is equal to AE, and CG is produced to H, so that GH is equal to CG. Show that HB, FB are in one straight line. Compare I. 16.

122. If one angle of a triangle is equal to the sum of the other two, the triangle can be divided into two isosceles triangles. I. 23.

123. The straight lines bisecting the angles at the base of an isosceles triangle meet the sides in D and E; show that DE is parallel to the base. I. 39.

124. In the figure of I. 5, if the equal sides of the triangle be produced upwards, I. 15 may be proved without assuming any proposition beyond I. 5.

125. The sum of the squares on the sides of a rhombus is equal to the sum of the squares on its diagonals.* I. 47.

126. Given the perpendicular of an equilateral triangle, construct it. I. 23.

127. Find a point such that the perpendiculars from it on two given straight lines shall be respectively equal to two given lengths. How many such points are there? I. 31.

128. If straight lines be drawn from the angles of a parallelogram perpendicular to a straight line outside the parallelogram; the sum of the perpendiculars from one pair of opposite angles is equal to the sum of the other two. Ded. 89.

129. If the angle C of a triangle be equal to the sum of the other two angles A and B, the side AB is equal to twice the straight line joining C to the mid point of AB. Ded. 122.

130. If any point be taken within a parallelogram, the sum of the triangles formed by joining the point with the ends of a pair of opposite sides, is half the parallelogram. I. 41.

131. $ABCD$ is a quadrilateral; construct a triangle whose base shall be in AB produced, vertex at a given point P in CD, and area equal to the quadrilateral. App. 18.

132. Construct a triangle of given area, and having two of its sides of given lengths. I. 41.

133. From a given point without the angle contained by two straight lines, draw a straight line, so that the part of it intercepted between the point and the nearest straight line, may be equal to the part between the two lines. App. 7.

134. If one diagonal of a quadrilateral bisects the other, it also bisects the quadrilateral. I. 38.

135. The straight line joining the mid point of the hypoteneuse of a right-angled triangle to the right angle, is equal to half the hypoteneuse. Ded. 122.

136. Upon a given hypoteneuse describe a right-angled triangle, one of whose sides shall be half the given base. Ded. 135.

137. If two straight lines be given in position, the locus of a point equidistant from them is a straight line. I. 26.

* We may assume that the square on a straight line = four times the square on half the line.

App. 7. 138. AB, CD, EF are three parallels. A, C, E are in a straight line, and so are B, D, F. If AC is equal to CE, prove that BD is equal to DF.

I. 34. 139. If $ABCD$ be a parallelogram, and from K a point in the diagonal AC, EKF be drawn parallel to AD, meeting AB and DC in E, and F; and HKG parallel to AB, meeting AD and BC in H, and G; the triangles AGF, AEH, are together equal to the triangle ABC.

I. 32. 140. If one angle of a triangle be triple another, the triangle may be divided into two isosceles triangles.

I. 4. 141. If in the sides of a given square at equal distances from the four angles, four other points be taken, one on each side, the figure formed by the lines joining them, is also a square.

I. 15. 142. ABC is a triangle; BD, CE, lines drawn making equal angles with BC, and meeting the opposite sides in D and E, and each other in F; prove that if the angle AFE is equal to the angle AFD, the triangle is isosceles.

App. 3. 143. Through two given points on opposite sides of a given straight line, draw two straight lines which shall meet in that straight line, and include an angle bisected by that line.

Ded. 135. 144. From the angle A of a triangle ABC, AD is drawn perpendicular to BC; and from B, BE perpendicular to AC (the sides being produced if necessary); if F be the middle point of AB, show that FD, FE are equal.

I. 4. 145. On the sides of any triangle ABC, equilateral triangles, BCD, CAE, ABF, are described, all external; show that the straight lines AD, BE, CF are all equal.

I. 47. 146. If any point P be taken inside a rectangle $ABCD$, the squares on PA and PC are together equal to the squares on PB and PD.

I. 32. 147. In the figure of I. 18 the angle LMN is equal to half the difference of the angles KLM and KML.

I. 41. 148. From the point D of a parallelogram $ABCD$, draw DFG meeting BC at F, and AB produced at G, and join AF, CG; show that the triangles ABF, CFG are equal.

App. 18. 149. ABC is a triangle; construct a triangle of equal area having its vertex at D in BC, and its base in the same straight line as AB.

App. 7. 150. ABC is a triangle, CD, BE, parallel lines meeting AB and AC produced in D and E; prove that if the triangles BCE, ACB are equal, D is the middle point of AB.

I. 34. 151. Find a point in the hypoteneuse of a right-angled triangle, such that the sum of the perpendiculars from it on the other two sides of the triangle may be equal to a given line. Between what limits must the latter line lie?

I. 38. 152. If the diagonals AC, BD of a quadrilateral intersect in O, and the triangle ABC is equal to twice the triangle ADC, prove that OB is equal to twice OD.

153. Construct a triangle equal in area to a given heptagon. App. 18.

154. Three straight lines meet in a point; draw another straight line cutting them, so that the intercepted segments may be equal. App. 7

155. If two exterior angles of a triangle be bisected, and from the point of intersection of the bisecting lines a line be drawn to the opposite angle of the triangle, it will bisect that angle. I. 47.

156. Of all parallelograms which can be formed with diameters of given lengths, the rhombus is the greatest. I. 19.

157. If two sides of a trapezium be parallel, its area is equal to half that of a parallelogram whose base is the sum of these two sides, and altitude the perpendicular distance between them. App. 25.

158. In the figure of I. 47, prove that BG is parallel to CH. I. 28.

159. From a given isosceles triangle, cut off a trapezium which shall have the same base as the triangle, and its three remaining sides all equal to each other. Ded. 123.

160. AB is the hypoteneuse of a right-angled triangle: find a point D in AB, such that DB may equal the perpendicular from D on AC. I. 9.

161. From the angles B, and C, of a triangle perpendiculars BE, CF are drawn to the sides AC, AB; show that EF is bisected by a perpendicular drawn to it, from the mid point of BC. Ded. 135.

162. The straight line which joins the mid points of the diagonals of a quadrilateral, which has two sides parallel, is parallel to those sides. I. 38.

163. In the figure of I. 1, if the circles meet in O and P, and KM produced meets one of the circles in A, then AOP is an equilateral triangle. I. 32.

164. Inscribe a parallelogram in a given triangle, so that its diagonals shall intersect at a fixed point. App. 10.

165. Determine the locus of a point, whose distance from a given point is equal to its distance from another given point. I. 11.

166. Draw a straight line, equal to one straight line, parallel to another, and terminated by two given straight lines. I. 34.

167. The sum of the squares on the sides of an equilateral triangle is equal to four times the square on the perpendicular from an angle on the opposite side. I. 47.

168. Describe a square equal to the difference of the squares on two given lines. Ded. 136.

169. Two straight lines AB, CD intersect in E, and the triangle AEC is equal to the triangle BED; show that BC is parallel to AD. I. 39.

170. If the sides of a regular pentagon be produced to meet, the angles formed by them are together equal to two right angles. I. 32.

171. No straight line can be placed within a parallelogram greater than the greater diameter. I. 19.

I. 32. 172. On a given base describe a triangle, whose angles shall be in the ratio of 3 : 4 : 5.

Compare I. 16. 173. The two sides of a triangle are together greater than twice the straight line drawn from the vertex to the mid point of the base.

I. 31. 174. If a quadrilateral have two of its opposite sides equal, and the other two parallel, but not equal, its opposite angles are together equal to two right angles.

I. 18. 175. Show that a scalene triangle cannot be divided into two parts which will coincide.

App. 16. 176. Inscribe a square of given magnitude in a given square.

I. 32. 177. Construct an isosceles triangle having each angle at the base one-fourth of the vertical angle.

I. 38. 178. If two triangles have two sides of the one equal to two sides of the other, each to each, and the sum of the angles contained by these sides equal to two right angles, the triangles are equal in area.

Ded. 174. 179. If one of the straight lines which join the ends of two equal straight lines towards the same parts, make the interior angles on the same side equal to each other, the joining lines shall be parallel.

App. 8. 180. The line drawn from the vertex of a triangle bisecting the base, also bisects every line parallel to the base.

I. 34. 181. If through a point O within a parallelogram $ABCD$ two straight lines are drawn parallel to the sides, and the parallelograms OB and OD are equal, then O lies on the diagonal AC.

I. 37. 182. Construct a rhombus equal to a given parallelogram.

I. 37. 183. Given the base and area of a triangle, find the locus of the vertex.

I. 33. 184. In Ded. 109, prove that the parallelogram is equal to half the difference of the triangles.

I. 38. 185. AD is drawn from the vertex of an isosceles triangle ABC, perpendicular to the base, and is bisected in E. If BE produced meet AC in F, prove that the triangle BCF is double of the triangle BAF.

I. 32. 186. If the three sides of one triangle be respectively perpendicular to the three sides of another, the triangles are equiangular.

I. 47. 187. Given the base of a triangle, and the difference of the squares on its sides, find the locus of its vertex.

I. 34. 188. Find the locus of a point which is always equidistant from a given straight line.

I. 32. 189. If in a right-angled triangle, the square on one of the sides containing the right angle be equal to three times the square on the other, one angle of the triangle is double another.

I. 34. 190. In the figure of I. 43, the triangle BKD is equal to the difference of the parallelograms GF and EH.

C.

191. If any point be taken within an equilateral triangle, the sum of the perpendiculars from it on the sides is constant.

192. Given the mid points of the sides of a triangle, construct the triangle.

193. Given the centres of the escribed circles of a triangle, construct the triangle.

194. If two adjacent sides of a parallelogram be given in length, prove that the diagonal through their intersection increases, as the angle between them decreases.

195. In any triangle, the square on the side subtending an acute angle is less than the squares on the other two sides.

196. In an obtuse angled triangle, the square on the side subtending the obtuse angle is greater than the squares on the other sides.

197. AB, AC are two given straight lines of unlimited length. Find two points P and Q in them, such that if PQ be joined, AP and PQ may be of given length, and contain a given angle.

198. AHK is an equilateral triangle ; $ABCD$ a rhombus whose sides are each equal to those of the triangle, and of which BC and CD pass through H and K respectively : show that the angle DAB is ten-ninths of a right angle.

199. Describe a triangle equal in area to a parallelogram, and having one angle common to both.

200. Construct an equilateral triangle equal to a regular hexagon.

201. If two sides of a triangle are given, the area is a maximum when they contain a right angle.

202. In I. 47 prove that AE is perpendicular to BK.

203. In I. 47 prove that the triangle KCE is equal to the triangle DBF.

204. If one square is equal to another square, a side of the first is equal to a side of the second.

205. If two adjacent corners of a rhombus be fixed, the loci of the other corners are two circles ; but if two opposite corners be fixed, the locus of the other corners is a straight line.

206. If the squares on the first and third sides of a quadrilateral be equal to the squares on the second and fourth, the diagonals are at right angles.

207. Construct a right-angled triangle, given the hypoteneuse and sum of the sides.

208. Construct a right-angled triangle, given the hypoteneuse and difference of the sides.

209. Construct a right-angled triangle, given the perimeter and an angle.

210. Within a parallelogram inscribe a rhombus, with one of its angles at a given point in a side of the parallelogram.

211. A is a given point, and B a given point in a given straight line ; draw from A a straight line AP to the given straight line, so that AP, PB may be of given length.

212. A straight line EA bisects the right angle of a triangle ABC, and ED bisects BC at right angles; show that DE is equal to DA.

213. Construct a right-angled triangle, given the hypoteneuse and the foot of the perpendicular on it from the right angle.

214. The difference of the base angles of any triangle, is double of the angle contained by a line drawn from the vextex perpendicular to the base, and another bisecting the angle at the vertex.

215. Find a point within a triangle from which if lines be drawn to the angles, they will trisect the triangle.

216. In I. 47 prove that if FG, KH meet in M, and MA, BC in N, that AN, BK, CF are concurrent.

217. The angle-bisectors of a parallelogram form a rectangular parallelogram, whose diagonals are parallel to the sides of the former.

218. The vertical angle C of the isosceles triangle ABC is half a right angle, and the perpendiculars AD, BE, let fall from A, B, on the opposite sides intersect in F. Show that FE is equal to EC.

219. Construct a triangle of given altitude, equal to a given triangle.

220. In the base BC of a triangle ABC any point D is taken; draw a straight line such that if the triangle ABC be folded along this line, the point A shall fall on the point D.

221. In I. 47, prove that the squares on EK and DF are together equal to five times the square on BC.

222. Of all triangles having the same vertical angle, and whose bases pass through the same point, the least is that whose base is bisected in that point.

223. If one of the sides of an isosceles triangle be bisected in D, and doubled by being produced through the extremity of the base to E; then the distance of the other extremity of the base from E is double that from D.

224. ABC is a triangle with BA greater than CA; the angle A is bisected by AD, meeting BC in D: show that BD is greater than CD.

225. AL and AM are two given straight lines, and P a given point between them; through P draw a straight line to form with AL and AM the minimum triangle.

226. Two triangles are on equal bases and between the same parallels; show that if a straight line be drawn parallel to the bases, the parts of it intercepted by the triangles are equal.

227. Construct a right-angled triangle, given one side, and the sum of the other side and the hypoteneuse.

228. On the sides AB, AC of a triangle, parallelograms $ABDE$, $ACFG$ are described; DE, FG meet in H; prove that the area of these parallelograms together, is equal to the area of the parallelogram on BC whose side is equal and parallel to AH.

229. Construct a triangle, given one side, and the directions of the lines from its ends which bisect the opposite sides.

230. Draw a parallel to the base of a triangle, so that the sum of the lower segments may be of given length.

231. Construct a right-angled triangle having given one side, and the difference of the other side and the hypoteneuse.

232. Prove I. 19 by a direct demonstration.

233. In a triangle describe a rectangle one of whose sides shall be parallel to the base, and the sum of two adjacent sides equal to a given straight line.

234. Trisect a triangle by lines through a point in one side.

235. Through two given points draw parallel lines, to cut two given parallel lines so as to form a rhombus.

236. If A be the right angle of a triangle ABC, and AC be double AB, then the angle B is more than double the angle C.

237. Given the base, difference of base angles, and sum of sides of a riangle, construct it.

238. Given the base, difference of base angles, and difference of sides of a triangle, construct it.

239. Trisect a parallelogram by lines drawn through an angular point.

240. If two angle-bisectors of a triangle are equal, the triangle is isosceles.

BOOK II.

A.

II. 2. 1. AB is a straight line, bisected at C, and a point D is taken in AC. Prove that the rectangles $CA . AD$. and $BC . CD$ are together equal to the square on half the line.

I. 31, 41. 2. ABC is a triangle, and AD the perpendicular from A on BC; prove that the area of the triangle is half the rectangle $BC . AD$.

II. 1. 3. AB is a straight line, and C, D points in it. Prove that the rectangle $AC . DB$ and the square on AB, are together equal to the rectangles $AD . BC$, $DB . BA$, and $CA . AB$.

I. 3.
II. 6. 4. AB is a straight line and C a point in it. Prove that the difference of the squares on AB and BC is equal to the rectangle contained by AC and the sum of AB, BC.

I. 3.
II. 5. 5. The rectangle contained by the sum and difference of two straight lines, is equal to the difference of their squares.

I. 3.
II. 6. 6. AB is a straight line, bisected at C, and produced through A to D, and through B to E; prove that the difference of the squares on CD, and CE, is equal to the rectangle contained by DE and the difference of AD, BE.

II. 7. 7. If a straight line AB be divided at C so that the rectangle $AB . BC$ is equal to the square on AC, prove that the squares on AB and BC are equal to three times the square on AC.

II. 10. 8. In the figure of II. 10, prove that the rectangle contained by $EC . DG$ is equal to the rectangle contained by $CB . BD$.

II. 12, 13. 9. If the lengths of the sides of a triangle are respectively 2, 3, and 4 feet, is it acute, right, or obtuse angled?

II. 14. 10. Describe an isosceles right-angled triangle equal to a given rectilineal figure.

I. 3.
II. 10. 11. AB is a straight line, and C, D points in it equidistant from the ends; prove that the squares on AB, CD are together double of the squares on DB, BC.

II. 5. 12. Describe a rectangle equal to the difference of two given squares.

II. 9. 13. Prove that the square on the sum of two straight lines, together with the square on their difference, is double of the squares on the two lines.

14. Two points, C and D, are taken in the diameter AB of a circle equidistant from the centre, and any point E in the circumference is joined to them; show that the squares on EC, ED are together equal to the squares on AC, AD. App. 24.

15. AB is a straight line produced through B to C, and through A to D, so that BC is equal to AD; prove that the squares on CD and AB are together equal to twice the squares on BD and BC. I. 3. II. 9.

16. In II. 11 prove that the rectangle contained by AH and the sum of AB, AH, is equal to the square on AB. II. 3, 11.

17. ABC is an isosceles triangle, with AB equal to AC; and AB is produced to D, so that BD is equal to AB; prove the square on CD is equal to the square on AB, together with twice the square on BC. App. 24.

18. AB is a straight line produced through B to C, and D is a point in AB. Prove that the rectangle $AB \cdot BC$, together with the square on AB, is equal to the rectangles $AC \cdot AD$, $AB \cdot BD$, and $DB \cdot BC$. II. 1.

19. If ABC be a triangle, and AD, the perpendicular on BC, be equal to BC, the squares on AB, AC, together with twice the rectangle $BD \cdot DC$, are equal to three times the square on AD. I. 47. II. 4.

20. ABC is a triangle, and $ABDE$ the square on AB. Prove that the sum of the squares on CB, CE is equal to the sum of those on CD, CA. I. 4, 6. App. 24.

21. In II. 11 prove that the rectangle $FC \cdot HB$ is equal to the rectangle $AC \cdot CK$. II. 11.

22. If AB be divided equally at C, and unequally at D, then the squares on AD, DB are together equal to twice the rectangle $AD \cdot DB$ with four times the square on DC. II. 9, 5.

23. AB is a straight line produced through B to C, and through A to D; prove that the rectangle contained by AC and BD is equal to the sum of the rectangles $AB \cdot BC$, $AB \cdot AD$, $AD \cdot BC$, together with the square on AB. II. 1.

24. The sum of the squares on the sides of a parallelogram is equal to the sum of the squares on the diagonals. App. 10, 24 II. 4, Cor.

25. From two angular points of an acute-angled triangle ABC, perpendiculars AD, BE are let fall on the opposite sides. Shew that the rectangle $AC \cdot CE$ is equal to the rectangle $BC \cdot CD$. II. 13.

26. Construct a rectangle equal to a given square, and with the sum of its two adjacent sides of given length. App. 30.

27. ABC is a triangle, right-angled at C, and D a point in BC; prove that the sum of the squares on AD, BC, is equal to the sum of the squares on AB, CD. II. 4, 12.

28. ABC is a triangle, D the mid point of AB, and E any other point in AB; prove that the squares on AC, CB, and twice the square on DE are equal to the squares on AE, EB, and twice the square on CD. II. 9. App. 24.

II. 4. 29. AB is a straight line divided into two parts at C, so that BC is less than AC; CE is taken in CA equal to CB. Prove that the square on the whole line is equal to twice the rectangle $AE.EB$ together with the square AE, and four times the square on BC.

I. 20. II. 13. 30. If the lengths of two sides of an acute-angled triangle be 3 inches and 4 inches respectively, between what limits must the length of the third side lie?

I. 12, 47. 31. Find a point in the base of a triangle, which divides it so that the difference between the squares on the segments of the base is equal to the difference between the squares on the sides.

I. 32, 6. II. 3. 32. The angle B of the triangle ABC is half a right angle; CE is drawn perpendicular to AB; prove that the difference of the squares on BC, CE is equal to the difference of the rectangles $AB.CE$ and $AE.EB$.

I. 47. II. 5. 33. The square on any straight line drawn from the vertex of an isosceles triangle to the base, is less than the square on a side of the triangle, by the rectangle contained by the segments of the base.

B.

41. If any point D be taken in the hypoteneuse AC of an isosceles right-angled triangle ABC, the squares on AD, DC are equal to twice the square on BD. II. 9.

42. The diagonals AC, BD, of a square intersect in O; through O a straight line is drawn meeting AD, BC in E and F. Prove that the rectangles contained by $AD . DE$, and $CB . BF$ are together equal to the square. II. 3.

43. ABC is an equilateral triangle, CD the perpendicular on AB, and E any point in AB produced; prove that the square on EC is equal to the square on EB together with the rectangle $BC . AE$. II. 6.

44. ABC is a triangle and AD a perpendicular from A on the base BC, or the base produced; in DC, or DC produced, DE is taken equal to DB. Prove that the difference of the squares on the sides of the triangle is equal to the rectangle $BC . CE$. II. 5.

45. In II. 11, prove that CH produced passes through the point of intersection of FG and DB produced. I. 43.

46. In II. 4, if O be the mid point of BD, prove that the squares on DG, GB are together equal to the square on AB and twice the square on OG. II. 9.

47. AB, AC are the sides of an isosceles triangle; AE is drawn perpendicular to AB, meeting BC produced in E, and AD is drawn perpendicular to BC. Prove that the difference of the squares on AE and AC is equal to twice the rectangle $DC . CE$ together with the square on EC. II. 6.

48. If a point P be joined to the four angular points A, B, C, D, of a rectangle, the squares on PA and PC are together equal to the squares on PB and PD. App. 24.

49. Divide a straight line so that the rectangle under its segments may be equal to a given rectangle. II. 14.

50. D is the mid point of AC, one of the sides of an equilateral triangle ABC. Prove that the square on BD is three-fourths of the square on BC. II. 4, Cor.

51. ABC is a triangle right-angled at C; D and E are points in AB, and AB produced, such that BD, BE, BC, are all equal. Show that the rectangle $AD . AE$ is equal to the square on AC. II. 6.

52. In the figure of II. 8, if O be the mid point of EK, prove that the squares on ED and DK are equal to the square on AB, together with twice the square on OD. II. 10.

53. Describe a rectangle equal to a given square, and having one of its sides equal to a given line. II. 14.

54. In II. 11, prove that the rectangle contained by the sum and difference of the parts, is equal to the rectangle contained by the parts. Ded. 5.

App. 24. 55. $ABCDE$ is a straight line divided so that AB, BC, CD, DE, are all equal, and O is an external point. Prove that the difference between the sum of the squares on OA, OE, and the sum of the squares on OB, OD is equal to six times the square on AB.

II. 11. 56. To a given straight line apply a rectangle whose area shall be equal to the difference of the squares on its sides.

II. 1. 57. Prove geometrically the Algebraic identity

$$(a + b)(c + d) = ac + bc + ad + bd.$$

II. 12. 58. If a straight line be drawn from one of the acute angles of a right-angled triangle to the mid point of the opposite side; the square on that line, and three times the square on half the bisected side, are together equal to the square on the hypoteneuse.

II. 13. 59. ABC is a triangle right-angled at C; and DE is drawn from a point D in AC, perpendicular to AB; show that the rectangle $AB \cdot AE$ is equal to the rectangle $AC \cdot AD$.

II. 6. 60. ABC is a triangle, right-angled at A; prove that the square on AB is equal to the rectangle contained by the sum and difference of AC, BC.

II. 14. 61. Construct a rectangle equal to a given square, and having a given perimeter.

II. 9. 62. The least square which can be described in a given square has an area equal to half the given square.

II. 7. 63. Prove that the rectangle contained by the diagonals of the squares on the whole line, and one of the parts, is equal to twice the rectangle contained by the whole line and that part.

I. 43. 64. In II. 9, prove that the rectangle $GF \cdot CL$ is equal to the rectangle $AC \cdot GL$.

II. 12. 65. Describe an obtuse-angled triangle such that the square on the side opposite to the obtuse angle may be greater than the sum of the squares on the sides containing the angle, by the rectangle contained by these sides.

Book I. Ded. 135. 66. If a perpendicular be drawn from the right angle of a triangle to the hypoteneuse, the square on it is equal to the rectangle contained by the segments of the base.

II. 9. 67. In II. 4, if the squares on AC, CB are equal to twice the rectangle $AC \cdot CB$, the line AB is bisected.

Prove this Algebraically.

II. 12. 68. Describe a triangle such that the square on one side shall be equal to the square on the other side, together with three times the square on the third.

II. 4. 69. In II. 11, prove that the square on AB and BC together, is equal to five times the square on AC.

II. 12. 70. If C be the obtuse angle of a triangle ABC, D and E the feet of the perpendiculars from A and B on the opposite sides, prove that the square on AB is equal to the sum of the rectangles $BC \cdot BD$ and $AC \cdot AE$.

71. ABC is an isosceles triangle, and from one end B of the base BC a perpendicular BD is let fall on AC; prove that the square on BC is equal to twice the rectangle $AC \cdot CD$. II. 7.

72. On a given straight line describe a rectangle equal to a given rectilineal figure. Ded. 53.

73. In II. 10, EC is produced to meet AG in K. Prove that the rectangle $AC \cdot DG$ is equal to the rectangle $AD \cdot CK$. I. 43.

74. In the figure of II. 4, if AG be joined and produced to meet BE in L, the rectangle $EK \cdot KL$ is equal to the square on BC. I. 31.

75. The difference between the squares on the sum and difference of two straight lines is equal to four times the rectangle contained by the lines. II. 5.

76. ABC is an acute-angled triangle; BE and CF are perpendiculars on the opposite sides. Prove that the square on BC is equal to the sum of the rectangles $AB \cdot BF$ and $AC \cdot CE$. II. 13.

77. ABC is an acute-angled triangle, CD the perpendicular on AB, and E the middle point of AD; prove that the square on AC and twice the square on BE are together equal to the squares on CB, BA, and twice the square on DE. II. 13.

78. In II. 11, if FK meet AB in Z, prove that HZ is equal to HB. Ded. 21.

79. $ABCD$ is a parallelogram, and the diagonals AC, BD intersect in O; on AC points F and G are taken such that $OF = OG$; prove that the squares on two adjacent sides of the parallelogram are equal to the squares on BF, BG, together with twice the rectangle $AF \cdot FC$. II. 5.

80. Produce a straight line AB to C, so that the sum of the squares on AB, AC may be equal to twice the rectangle $AC \cdot CB$. II. 12.

81. If AB be divided at C so that the rectangle $AB \cdot BC$ is equal to the square on AC, and CD be taken equal to BC, then the rectangle $AC \cdot AD$ is equal to the square on CD. II. 2.

II. 13.

82. If DE be drawn parallel to the base BC of an isosceles triangle ABC, the square on BE is equal to the square on CE together with the rectangle $BC \cdot DE$.

83. Describe an isosceles triangle such that the square on the base may be equal to three times the square on either of the sides. II. 12.

84. Divide a straight line into two parts, so that the rectangle contained by the whole and one part may be equal to the rectangle contained by the other part and another given line. II. 14.

85. The rectangle contained by the diagonals of the squares on the sides of a rectangle is double the latter rectangle. II. 4.

86. In II. 11, prove that HD is parallel to FK. Ded. 78.

Ded. 33. 87. Produce one side of a given triangle, so that the rectangle contained by this side and the produced part may be equal to the difference of the squares on the other two sides.

II. 11. 88. Produce a straight line so that the rectangle contained by the whole straight line thus produced, and the part produced, may be equal to the square on the given line.

I. 43. 89. If ABC, DEF be two equiangular triangles, right-angled at B and E respectively, prove that the rectangle $AB \cdot EF$ is equal to the rectangle $BC \cdot DE$.

Ded. 45. 90. In II. 11, if FD meet AB in X, and GK in Y, prove that FX is equal to DY.

I. 19. 91. Prove from II. 14, that of all rectangles with a given perimeter the square has the greatest area.

C.

101. If AB be divided at C so that the square on AC is double the square on CB, the sum of AB and BC will be equal to the diameter of the square on AB.

102. If two rectangles are equal in area and perimeter, they are equal in every respect.

103. $ABCDE$ is a straight line divided so that AB, BC, CD, DE are all equal, and O is an external point. Prove that the difference of the squares on OA and OE is equal to twice the difference of the squares on OB, OD.

104. If the straight line PQ is divided at R so that the rectangle $PQ \cdot QR$ is equal to the square on PR, and PR is divided at S so that the rectangle $PR \cdot RS$ is equal to the square on PS, prove that PS is equal to RQ.

105. In any quadrilateral the squares on the diagonals are together double of the squares on the lines joining the middle points of opposite sides.

106. Divide a straight line into two parts so that the sum of their squares may be double the square on a given line.

107. Produce a straight line so that the sum of the squares on the whole line thus produced, and on the part produced, may be double the square on the given line.

108. In II. 11, if CH be produced to meet BF at O, shew that CO is at right angles to BF.

109. Construct a rectangle, given its area, and the difference of the squares on its sides.

110. Produce a given straight line so that the rectangle contained by the whole line thus produced, and the part produced, may be equal to the square on half the line.

111. In II. 11, if BE and CH meet in X, then AX is perpendicular to CH.

112. AB is the hypoteneuse of a right-angled triangle ABC; from it AD, BE are cut off equal to AC and BC respectively; shew that the square on DE is equal to twice the rectangle $AE \cdot BD$.

113. Divide a straight line into two parts, such that the rectangle contained by one of them, and another straight line, may be equal to the square on the remaining part.

114. In II. 11, prove that GB, DF, AK are parallel.

115. Describe a right-angled triangle, such that the rectangle contained by the hypoteneuse and one side may be equal to the square on the other side.

116. Divide a straight line into two parts, such that their rectangle may be equal to the square on their difference.

117. In the figure of I. 43 prove that the rectangle contained by AH and EB is equal to the rectangle contained by AE and HD.

118. AC, BD, the diagonals of a square meet in O, and are produced to E and F respectively. Through O, GOH is drawn parallel to the

side AD or BC; the angle CDF is bisected by DK meeting GH produced in K, and HK is produced to L, making KL equal to KH. Prove that twice the rectangle $GH \,.\, HL$ is equal to the difference between the square on HL and the given square.

119. Divide a straight line into two parts, so that the squares on the whole line and on one of the parts may be together double of the square on the other part.

120. In II. 9, if AF meet EC in L, prove that the rectangle $AL \,.\, LG$ is equal to the rectangle $FL \,.\, LC$.

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